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The Price of Risk and Ambiguity in an Intertemporal General Equilibrium Model of Asset Prices*

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Abstract

We consider a version of the intertemporal general equilibrium model of Cox et al. (1985a) with a single production process and two correlated state variables. It is assumed that only one of them, Y_2, has shocks correlated with those of the economy's output rate and, simultaneously, that the representative agent is ambiguous about its stochastic process. This implies that changes in Y_2 should be hedged and its uncertainty priced, with this price containing risk and ambiguity components. Ambiguity impacts asset pricing through two channels: the price of uncertainty associated with the ambiguous state variable, Y_2, and the interest rate. With ambiguity, the equilibrium price of uncertainty associated with Y_2 and the equilibrium interest rate can increase or decrease, depending on the relation between (i) the correlations between the shocks in Y_2 and those in the output rate and in the other state variable; (ii) the diffusion functions of the stochastic processes for Y_2 and for the output rate; and (iii) the gradient of the value function with respect to Y_2. As applications of our generic setting, we deduct the model of Longstaff and Schwartz (1992) for interest-rate-sensitive contingent claim pricing and the variance risk price specification in the option pricing model of Heston (1993).

Keywords: Ambiguity, Asset Pricing, Equilibrium Price of Uncertainty.

JEL Classification: C68 · D81 · G13

1 Introduction

There are two major approaches for the modelling of asset prices and of the implied uncertainty prices: the equilibrium and the arbitrage approach.

The equilibrium approach includes models that start by describing the production sector of the economy, which is typically a set of production processes driven by exogenous state variables whose dynamics are, in turn, described by stochastic processes. The assets are contingent claims to the output of these production processes. With the objective of maximizing an utility function, the representative agent decides how much to consume and how much to invest (either physically in the production processes or financially by acquiring assets). The equilibrium prices of the assets, and the corresponding

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uncertainty prices, must be such that demand equals supply. An example of this type of setting is the continuous time model of Cox et al. (1985a), which has several applications in the literature (e.g. Cox et al. (1985b), Longstaff and Schwartz (1992) and Gagliardini et al. (2009) on the modeling of the term structure of interest rates). A pioneering example of the equilibrium approach is the model of Lucas (1978), which has a similar structure to that of Cox et al. (1985a), but has exogenous production, no technological change and is in discrete time. The model of Lucas (1978) has also several applications in the literature, with a recent example being the consumption based general equilibrium model for designing affine asset pricing models by Eraker and Shaliastovich (2008).

The arbitrage approach starts by assuming the dynamics of the state variables, of which the contingent claims depend, and an exogenous specification for the uncertainty prices. Then, by applying Itô's lemma and imposing the condition that there are no arbitrage opportunities, the prices of contingent claims are obtained. This is the standard approach in the option pricing literature (e.g. Black and Scholes (1973)), with some exceptions (e.g. Amin and Ng (1993)), and has been extensively used in other fields of finance. For example, Vasicek (1977) and Brennan and Schwartz (1979) used it to model the term structure of interest rates.

The equilibrium approach has clear advantages with respect to the arbitrage approach. As pointed out by Cox et al. (1985b), imposing exogenous uncertainty prices without any underlying economic equilibrium may lead to internal inconsistencies. In the equilibrium approach, uncertainty prices are endogenous and therefore part of the equilibrium. Moreover, models under the arbitrage approach say very little about the economic nature of the price of uncertainty.

In this paper, we consider a continuous time general equilibrium model for contingent claim pricing which is a two state variable version of the model of Cox et al. (1985a). It is assumed that the two state variables, Y_{1t} and Y_{2t}, are correlated and both impact the expected return of the single production process, Q_t. Moreover, it is assumed that shocks in one of the state variables, Y_{2t}, are correlated with those in the return of the production process, and that the representative agent is ambiguous about the stochastic process describing the dynamics of Y_{2t}. Uncertainty in the model has therefore two dimensions: risk and ambiguity.[1]

Ambiguity about the stochastic process for the state variable Y_{2t} is introduced through a robust control approach.[2] The representative agent considers contaminations, P^h, around a reference belief model, P. Aversion towards ambiguity is considered by assuming that, in the spirit of Gilboa and Schmeidler (1989), the agent chooses the worst possible contamination, i.e., the one associated with the lowest expected utility.[3] It is found that ambiguity about Y_{2t} impacts the fundamental partial

[1]The distinction between risk and ambiguity was first pointed out by Knight (1921) and later supported by the empirical experiments of Ellsberg (1961) and others (see Camerer and Weber (1992) and Epstein and Schneider (2010) for a survey). The reason for this distinction is that economic agents may not be able to completely describe the uncertainty that they face by using a single probability distribution. Risk refers to uncertainty that can be represented by a probability distribution, while ambiguity refers to uncertainty that cannot. This distinction has relevant implications for the behavior of economic agents, and, therefore, for economic theory in general. That is why a rapidly growing literature on asset pricing under ambiguity aversion is emerging. This literature has been comprehensively surveyed by Epstein and Schneider (2010).

[2]An extensive review on decision theory under ambiguity has been carried out by Etner et al. (2009). Briefly, the two most common approaches being used in the ambiguity literature are: the robust control (RC) approach, associated to an assumption of model uncertainty (as, for e.g., in Maenhout (2004, 2006) and Gagliardini et al. (2009)); the multiple priors (MP) approach, from the seminal work by Gilboa and Schmeidler (1989), where the single probability measure of the standard expected utility model is replaced by a set of probabilities or priors. The relationship between the robust control and multiple priors approaches has been widely discussed in the literature, for e.g., in Hansen and Sargent (2001), Hansen et al. (2002), Epstein and Schneider (2003), and Maccheroni et al. (2006).

[3]The approach of Gilboa and Schmeidler (1989) is sometimes criticized because it apparently implies extreme ambiguity aversion. However, the implied decision criteria may not be so extreme as it seems. The reasoning for this is that the set of priors is not an independent object including all logically possible priors, being instead part of the representation of the concrete problem under analysis. This is why the criteria of Gilboa and Schmeidler (1989) is not so extreme as, for example, the Wald maxmin criteria. As Epstein and Schneider (2010) claimed: *"Ultimately, the only way to argue that the model is extreme is to demonstrate extreme behavioral implications of the axioms, something that has not been done"*. More recently, a smooth ambiguity aversion utility theory has been developed on the back of the seminal work of Klibanoff et al. (2005). It is claimed that this setup distinguishes ambiguity from ambiguity aversion and allows for smooth indifference curves, avoiding the infinite ambiguity aversion implied in the approach of Gilboa

differential equation satisfied by the price of a contingent claim through two channels: the equilibrium uncertainty price associated with the ambiguous state variable, Y_{2t}, and the equilibrium interest rate.

The specification for each of those channels, containing a risk and an ambiguity component, is obtained. Moreover, we conclude that the impact of ambiguity on the equilibrium price of uncertainty associated with Y_{2t}, and on equilibrium interest rate depends on: (i) the correlations between the shocks in Y_{2t} and the shocks in the other state variable and in the output rate; (ii) the diffusion functions of the stochastic processes of Y_{2t} and of the economy's output rate; and (iii) the impact on utility of changes in Y_{2t}.

The major contribution of this paper is to develop a two-factor general equilibrium framework for asset pricing under ambiguity when the shocks in the two state variables are correlated but only the shocks in the ambiguous state variable are correlated with those of the economy's output rate. This is a simple setting that can be applied to many asset pricing problems. As an example, we apply our general results to the specific investment opportunity set of the well known option pricing model of Heston (1993), therefore providing an equilibrium motivation for the specification of the price of variance risk used there.

The paper is organized as follows. In section 2, the intertemporal general equilibrium model for contingent claim pricing under ambiguity is developed. In section 3, it is applied to a concrete investment opportunity set which contains that of the option pricing model of Heston (1993). In section 4, we conclude the paper with some remarks.

2 General Equilibrium Framework

We consider an intertemporal general equilibrium model for contingent claim pricing that is a version of the model of Cox et al. (1985a) with two correlated state variables, a single stochastic constant returns-to-scale production process and logarithmic utility.[4]

There is a single physical good in the economy, that the representative agent can consume or reinvest in the stochastic production process, Q_t. The realized return on the physical investment made through the production process, i.e., the economy's output rate, is driven by two correlated state variables, Y_{1t} and Y_{2t}:

$$\frac{dQ_t}{Q_t} = g_Q\left(Y_{1t}, Y_{2t}\right) dt + \sigma_Q\left(Y_{1t}, Y_{2t}\right) dW_Q\,, \tag{1}$$

where $g_Q\left(Y_{1t}, Y_{2t}\right)$ and $\sigma_Q\left(Y_{1t}, Y_{2t}\right)$ are generic expressions for the drift and diffusion functions of the output rate dynamics, which may depend on both state variables, and W_Q is a standard Brownian motion.

The dynamics of the state variables, (Y_{1t}, Y_{2t}), is given by:

$$dY_{1t} = g_{Y_1}\left(Y_{1t}, Y_{2t}\right) dt + \sigma_{Y_1}\left(Y_{1t}, Y_{2t}\right) dW_1\,, \tag{2}$$

$$dY_{2t} = g_{Y_2}\left(Y_{1t}, Y_{2t}\right) dt + \sigma_{Y_2}\left(Y_{1t}, Y_{2t}\right) dW_2\,, \tag{3}$$

where the generic functions g_{Y_1}, σ_{Y_1}, g_{Y_2}, and σ_{Y_2} have the same meaning, for each of the state variables, as g_Q and σ_Q for the stochastic process (1).[5] The processes W_1 and W_2 are standard Brownian motions with an instantaneous correlation equal to ρ ($dW_1 dW_2 = \rho dt$).

and Schmeidler (1989). However, there is still a debate in the literature about the axiomatic foundations of this line of models (see Epstein (2010) and Klibanoff et al. (2009) for a recent exchange on this). Also because of this, the approach of Gilboa and Schmeidler (1989) continues to be the main reference in the literature.

[4]We foresee as interesting extensions of this setting the consideration of heterogeneous agents, where trade would occur in equilibrium, and of a more general setting for preferences (e.g. SDU from Duffie and Epstein (1992a,b)). We thank Frank Riedel for those suggestions for future work.

[5]Diffusion functions σ_Q, σ_{Y_1} and σ_{Y_2} are assumed to be positive, as it is usual in the literature, due to the analogy with the statistical concept of standard deviation, although probabilistically it is not required so. We thank Paolo Porchia for this insight.

Both state variables potentially impact the expected output rate, but it is assumed that only one of them, Y_{2t}, has shocks that are correlated with those of the output rate. The instantaneous correlation between $\frac{dQ_t}{Q_t}$ and dY_{2t} is ρ_2 ($dW_Q dW_2 = \rho_2 dt$).

The investment opportunity set given by (1), (2) and (3) can be described by the following system (Appendix 5.1):

$$\begin{bmatrix} \frac{dQ_t}{Q_t} \\ dY_{1t} \\ dY_{2t} \end{bmatrix} = \begin{bmatrix} g_Q \\ g_{Y_1} \\ g_{Y_2} \end{bmatrix} dt + \underbrace{\begin{bmatrix} \sigma_Q \sqrt{\left(1 - \frac{\rho_2^2}{1-\rho^2}\right)} & 0 & \frac{\sigma_Q \rho_2}{\sqrt{1-\rho^2}} \\ 0 & \sigma_{Y_1} & 0 \\ 0 & \sigma_{Y_2}\rho & \sigma_{Y_2}\sqrt{1-\rho^2} \end{bmatrix}}_{A} \begin{bmatrix} dZ_0 \\ dZ_1 \\ dZ_2 \end{bmatrix}, \quad (4)$$

where Z_i $(i = 0, 1, 2)$ are independent Brownian motions. We assume that ρ, $\rho_2 \in \,]-1, 1[$ (i.e., we exclude perfect correlations) and that $\rho_2 < \sqrt{1-\rho^2}$ to guarantee that the elements of the matrix A are real numbers. For the presentation that follows, we make use of the following three matrices, Z, σ and Ξ:

$$\begin{aligned} Z &= \begin{bmatrix} Z_0 \\ Z_1 \\ Z_2 \end{bmatrix}, \\ \sigma &= \begin{bmatrix} \sigma_Q \sqrt{\left(1 - \frac{\rho_2^2}{1-\rho^2}\right)} & 0 & \frac{\sigma_Q \rho_2}{\sqrt{1-\rho^2}} \end{bmatrix}, \\ \Xi &= \begin{bmatrix} 0 & \sigma_{Y_1} & 0 \\ 0 & \sigma_{Y_2}\rho & \sigma_{Y_2}\sqrt{1-\rho^2} \end{bmatrix}. \end{aligned} \quad (5)$$

Observe that $\Xi\Xi^\top$ represents the covariance matrix of the state variables (Y_{1t},Y_{2t}):

$$\Xi\Xi^\top = \begin{bmatrix} \sigma_{Y_1}^2 & \sigma_{Y_2}\sigma_{Y_1}\rho \\ \sigma_{Y_2}\sigma_{Y_1}\rho & \sigma_{Y_2}^2 \end{bmatrix}.$$

It is assumed that the representative agent is not totally sure about the data-generating processes (4) that characterize the investment opportunity set dynamics. This means that the uncertainty faced by the representative agent has two dimensions: risk and ambiguity.

Ambiguity about the investment opportunity set is introduced through a "constraint preferences" robust control approach, following the extension of the model of Cox et al. (1985b) made by Gagliardini et al. (2009).

It is assumed that the representative agent is ambiguous about the dynamics of Y_{2t}. The agent considers contaminations (alternative models), P^h, around his reference belief, P. The contaminations are assumed to be absolutely continuous with respect to P, and, therefore, are equivalently described by contaminating drift processes, h. In each of the alternative models, P^h, the Brownian motion becomes, therefore, $Z^h(t) = Z(t) + \int_0^t h(s)\, ds$.[6]

[6] Gagliardini et al. (2009) explain that, for tractability reasons, the analysis is restricted to the class of Markov-Girsanov kernels. The absolute continuity assumption between P and P^h guarantees the equivalence property between the probability measures and, consequently, that the Cameron-Martin-Girsanov theorem can be applied. Moreover, from this theorem and considering the diffusion family of models under consideration, all that a probability measure change implies is the change of the drift function of the stochastic processes.

Existence of ambiguity is analytically represented by perturbations of the drift, with respect to the reference belief, in the dynamics of the ambiguous state variable, Y_{2t}. Aversion towards ambiguity is introduced by assuming that, in the spirit of Gilboa and Schmeidler (1989), the representative agent chooses from all the possible contaminations, P^h, the one that corresponds to the worst case scenario, i.e., the one associated with lower expected utility.

An upper bound is imposed on the contaminating drift processes, h:

$$h^\top h \leqslant 2\eta, \tag{6}$$

where $\eta \geqslant 0$ is a parameter that can be interpreted as the level of ambiguity.

As highlighted by Gagliardini et al. (2009), the bound (6) should be such that alternative models are statistically close to the "reference belief" model: otherwise the agent would easily distinguish among them and, consequently, would not face ambiguity. That is, η should be small. Moreover, the bound (6) constrains both the instantaneous time variation and the continuation value of the relative entropy between the reference belief, P, and any admissible contaminated belief, P^h. Trojani and Vanini (2004) explain that the set $\left\{h : h^\top h \in [0\,,\,2\eta]\;,\;\forall t \geqslant 0\right\}$ defines a rectangular set of priors because any process h (and therefore any probability measure P^h) in this set corresponds to a selection of transition densities from t to $t+dt$, $t \geqslant 0$, such that $h^\top h \in [0\,,\,2\eta]$. The fact that the specification of the ambiguity aversion is based on a rectangular set of priors guarantees a dynamically consistent preference ordering, and can be interpreted as a continuous time version of Epstein and Schneider's (2003) Recursive Multiple Priors Utility.[7] More generally, in Hansen and Sargent (2006) there is a comprehensive discussion of the dynamic consistency issue under the robust control approach.

Considering the system (4) that describes the investment opportunity set dynamics, ambiguity about Y_{2t} is introduced through contaminations of the Brownian Motion Z_2. As in Gagliardini et al. (2009), for a two state-variable model, the admissible contaminating drift process is restricted to be $h = \begin{bmatrix} h_0 & h_1 & h_2 \end{bmatrix}^\top = \begin{bmatrix} 0 & 0 & h_2 \end{bmatrix}^\top$. The class of admissible Markovian drift contaminations satisfying this restriction and the entropy bound is denoted by $\mathcal{H}$.

Under an admissible contamination, P^h, the investment opportunity set is therefore described by:

$$\begin{bmatrix} \frac{dQ_t}{Q_t} \\ dY_{1t} \\ dY_{2t} \end{bmatrix} = \begin{bmatrix} g_Q \\ g_{Y_1} \\ g_{Y_2} \end{bmatrix} dt + \begin{bmatrix} \sigma_Q\sqrt{\left(1-\frac{\rho_2^2}{1-\rho^2}\right)} & 0 & \frac{\sigma_Q\rho_2}{\sqrt{1-\rho^2}} \\ 0 & \sigma_{Y_1} & 0 \\ 0 & \sigma_{Y_2}\rho & \sigma_{Y_2}\sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} dZ_0 \\ dZ_1 \\ dZ_2 + h_2 dt \end{bmatrix}. \tag{7}$$

Note that in the "contaminated" system (7) that describes the investment opportunity set dynamics, the diffusion component continues to be driven by the same vector of independent Brownian motions, Z in (5). It is also straightforward to observe that the contamination h_2 only perturbs the drift functions in the stochastic processes of $\frac{dQ_t}{Q_t}$ and dY_{2t}, while keeping unchanged their diffusion functions.

The intertemporal budget constraint faced by the agent is given by:

$$dW_t = W_t\frac{dQ_t}{Q_t} - C_t dt\,,$$

where W_t and C_t represent wealth and consumption at time t. Considering the output rate dynamics

[7] See Epstein and Schneider (2003) for the definition of the rectangularity property. Additionally, in Trojani and Vanini (2004), p. 289, there is a detailed explanation supporting the rectangularity property of the present set of priors built under the constraint (6), and how this rectangular set of priors can be defined in the *k-ignorance* model of Chen and Epstein (2002).

$\frac{dQ_t}{Q_t}$ in (7), the dynamic budget constraint can be expressed as:

$$dW_t = (W_t g_Q - C_t)\, dt + W_t \sigma \begin{bmatrix} dZ_0 \\ dZ_1 \\ dZ_2 + h_2 dt \end{bmatrix}, \tag{8}$$

with matrix σ disclosed in (5).

If the representative agent were not ambiguous about the dynamics of Y_{2t}, then his problem would be to find the optimal consumption strategy, $C : [0, +\infty[\to \mathbb{R}_+$, that maximizes his expected intertemporal utility. As in the setting of Cox et al. (1985a), the optimal consumption strategy is financed by allocating all the wealth in the production process and none in the financial assets (which are in zero net supply). This implies that the only relevant control variable, for the non-ambiguous agent, is the consumption flow process.

However, with the representative agent being ambiguous about the dynamics of Y_{2t}, there isn't a single probability measure, P, to be considered when assessing his expected utility. Instead, a set of probability measures, P^h, has to be considered. The existence of ambiguity therefore implies that the solution of the representative agent's problem also involves solving for the most adverse contaminating drift process $h \in H$.

Having a logarithmic instantaneous utility function, the ambiguity averse representative agent solves the following Maxmin expected utility program:

$$J(W_0, Y_{10}, Y_{20}) = \sup_{C} \inf_{h \in \mathcal{H}} E^h \left[\int_0^\infty e^{-\delta s} ln\,(C_s)\, ds \right], \tag{9}$$

subject to the dynamics of state variables Y_{1t} and Y_{2t}, represented in (7), and to the dynamic budget constraint (8). The operator E^h denotes expectations under the measure P^h, $\delta > 0$ is the subjective rate of discount of the representative agent, and $J(W_0, Y_{10}, Y_{20}) = J(W_t, Y_{1t}, Y_{2t})\mid_{t=0}$ denotes the value function of the problem.

Applying Proposition 1 in Gagliardini et al. (2009), the value function of the ambiguity-averse agent is given by:

$$J(W_0, Y_{10}, Y_{20}) = -\frac{1}{\delta} + \frac{ln\,(\delta W_0)}{\delta} + \frac{1}{\delta} V(Y_{10}, Y_{20}), \tag{10}$$

where

$$V(Y_{10}, Y_{20}) = \inf_{h \in \mathcal{H}} E^h \left[\int_0^\infty e^{-\delta s} \left(g_Q - \frac{1}{2}\sigma\sigma^\top + \sigma h_s \right) ds \right], \tag{11}$$

subject to dY_{1t} and dY_{2t} in (7) and with σ given by (5). The corresponding Bellman equation solved by the value function $V(Y_{10}, Y_{20})$ is given by:

$$\begin{aligned} 0 = {} & V_Y^\top g_Y + \frac{1}{2} trace\left[\Xi^\top V_{YY} \Xi\right] - \sqrt{2\eta}\sqrt{\left(\Xi^\top V_Y + \sigma^\top\right)^\top \left(\Xi^\top V_Y + \sigma^\top\right)} \\ & + g_Q - \frac{1}{2}\sigma\sigma^\top - \delta V, \end{aligned} \tag{12}$$

where: (i) V_Y and V_{YY} are the gradient and Hessian matrices of the value function $V(Y_{10}, Y_{20})$ with respect to the state variables;[8] (ii) σ and Ξ are the matrices in (5), with $\Xi\Xi^\top$ representing the covariance matrix of the state variables (Y_{1t},Y_{2t}); and (iii) g_Y is the vector of drift functions of the state variables.

[8] In general, it is known that a value function may not be differentiable, at least in the entire domain of the state variables. If the differentiability property is not satisfied then the "viscosity solution" of the stochastic optimal control problem has to be studied.

The equilibrium contamination drift vector, $h^* = \begin{bmatrix} 0 & 0 & h_2 \end{bmatrix}^\top$, that solves the model selection problem is obtained directly from Proposition 1 in Gagliardini et al. (2009), with equilibrium h_2 being given by (Appendix 5.2):

$$h_2 = \begin{cases} -\sqrt{2\eta} & \text{if } V_{Y_2} > -\frac{\sigma_Q \rho_2}{\sigma_{Y_2}(1-\rho^2)}, \\ \\ \sqrt{2\eta} & \text{if } V_{Y_2} < -\frac{\sigma_Q \rho_2}{\sigma_{Y_2}(1-\rho^2)}, \end{cases} \tag{13}$$

where V_{Y_2} represents the gradient of the value function $V(Y_{1t}, Y_{2t})$ in (11), with respect to the ambiguous state variable Y_{2t}. The validity of this expression is guaranteed by previous assumptions of $\rho \in]-1, 1[$ and $\sigma_{Y_2} > 0$ (non-deterministic state variable).

From the Cameron-Martin-Girsanov theorem, coupled with the fact that in our setting only diffusion models are considered, it results that the change from one probability measure to an equivalent probability measure only leads to a change of drift in the stochastic processes of the state variables. Considering the reference belief P and an equivalent "uncertainty-neutralized" probability measure, then the change of drift associated with each of the state variables, represented by the matrix $\phi = \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix}^\top$, is the equilibrium price of uncertainty associated to each of the state variables.

In the present setting, the equilibrium prices of uncertainty associated to Y_{1t} and Y_{2t} are given by (Appendix 5.2):

$$\phi_1 = 0, \tag{14}$$

$$\phi_2 = \begin{cases} \sigma_{Y_2}\sigma_Q\rho_2 + \sigma_{Y_2}\sqrt{2\eta}\sqrt{1-\rho^2} & \text{if } V_{Y_2} > -\frac{\sigma_Q \rho_2}{\sigma_{Y_2}(1-\rho^2)}, \\ \\ \sigma_{Y_2}\sigma_Q\rho_2 - \sigma_{Y_2}\sqrt{2\eta}\sqrt{1-\rho^2} & \text{if } V_{Y_2} < -\frac{\sigma_Q \rho_2}{\sigma_{Y_2}(1-\rho^2)}, \end{cases} \tag{15}$$

respectively.

The result in (14) means that, in equilibrium, uncertainty about Y_{1t} is not priced. This should not be a surprise, considering that there is no ambiguity about the dynamics of Y_1 and that its shocks are uncorrelated with those of the output rate. Regarding the state variable Y_{2t}, there exists an associated equilibrium price of uncertainty, given by (15), as its shocks are correlated with those of economy's output rate (implying an equilibrium price of risk) and the representative agent is ambiguous about its stochastic process (implying an equilibrium price of ambiguity).

In fact, from (15), it is clear that the equilibrium price of uncertainty associated with Y_{2t} is divided in two components: the equilibrium price of risk, given by $\sigma_{Y_2}\sigma_Q\rho_2$, and the equilibrium price of ambiguity, given by $\pm\sigma_{Y_2}\sqrt{2\eta}\sqrt{(1-\rho^2)}$.

We therefore conclude that the existence of ambiguity about Y_{2t} implies an additional equilibrium price component. From (15), it results that such component can be positive or negative. Consequently, the equilibrium uncertainty price associated with that state variable, Y_{2t}, can either increase or decrease when the agent is ambiguous about its stochastic process, depending on the relation between: (i) the impact on the indirect utility (value function) of changes in the ambiguous state variable (V_{Y_2}); (ii) the correlations between shocks in that variable and in the other state variable (ρ) as well as in the economy's output rate (ρ_2); and (iii) the diffusion functions of the stochastic processes for the ambiguous state variable (σ_{Y_2}) and the economy's output rate (σ_Q).

Note also that the uncertainty price becomes preference-dependent when ambiguity aversion is considered (as it includes the η parameter). This does not happen when uncertainty is exclusively risk.

In order to derive the partial differential equation (PDE) satisfied by the contingent claim price under the present setting, it is still necessary to obtain the equilibrium instantaneous interest rate, r_t.

The generic expression for r_t is given by (Appendix 5.2):

$$r_t = \begin{cases} g_Q - \sigma_Q^2 - \frac{\sigma_Q \rho_2}{\sqrt{1-\rho^2}} \sqrt{2\eta} & \text{if } V_{Y_2} > -\frac{\sigma_Q \rho_2}{\sigma_{Y_2}(1-\rho^2)} , \\ g_Q - \sigma_Q^2 + \frac{\sigma_Q \rho_2}{\sqrt{1-\rho^2}} \sqrt{2\eta} & \text{if } V_{Y_2} < -\frac{\sigma_Q \rho_2}{\sigma_{Y_2}(1-\rho^2)} , \end{cases} \tag{16}$$

where the first two parcels $\left(g_Q - \sigma_Q^2\right)$, give the equilibrium instantaneous interest rate when uncertainty is exclusively risk ($\eta = 0$), and the third parcel $\left(\pm \frac{\sigma_Q \rho_2}{\sqrt{1-\rho^2}} \sqrt{2\eta}\right)$ is the new component that results from the existence of ambiguity. The expression for the equilibrium interest rate when uncertainty is exclusively risk, given by the difference between the expected output rate (g_Q) and the variance of the output rate $\left(\sigma_Q^2\right)$, is consistent with findings in the literature based on the setting of Cox et al. (1985a), without ambiguity (e.g. in Longstaff and Schwartz (1992)).[9]

Moreover, looking at the ambiguity component in (16), we conclude that ambiguity about the stochastic process of Y_{2t} does not impact the equilibrium interest rate when the economy's output rate is deterministic ($\sigma_Q = 0$) or when shocks in the ambiguous state variable are uncorrelated with those of the output rate ($\rho_2 = 0$). In general, the equilibrium instantaneous interest rate under ambiguity, r_t, can be higher or lower than when uncertainty is exclusively risk, depending on the sign of ρ_2 and, as in (15), on the relation between: (i) the impact on the indirect utility (value function) of changes in the ambiguous state variable (V_{Y_2}); (ii) the correlations between shocks in that variable and in the other state variable (ρ) as well as in the economy's output rate (ρ_2); and (iii) the diffusion functions of the stochastic processes for the ambiguous state variable (σ_{Y_2}) and economy's output rate (σ_Q).[10]

Given the equilibrium price of uncertainty associated with Y_{1t} and Y_{2t}, (14) and (15) respectively, and the equilibrium instantaneous interest rate (16), from Proposition 2 in Gagliardini et al. (2009) the fundamental partial differential equation (PDE) satisfied by the price of a contingent claim with maturity τ, $H(Y_{1t}, Y_{2t}, \tau)$, assumed to depend on both state variables but not on wealth, is given by:

$$\begin{aligned} \frac{\partial H}{\partial \tau} = & \frac{1}{2}\sigma_{Y_1}^2 (Y_{1t}, Y_{2t}) \frac{\partial^2 H}{\partial Y_1^2} + \frac{1}{2}\sigma_{Y_2}^2 (Y_{1t}, Y_{2t}) \frac{\partial^2 H}{\partial Y_2^2} \\ & + \rho \sigma_{Y_1} (Y_{1t}, Y_{2t}) \sigma_{Y_2} (Y_{1t}, Y_{2t}) \frac{\partial^2 H}{\partial Y_1 \partial Y_2} + [g_{Y_1} (Y_{1t}, Y_{2t}) - \phi_1] \frac{\partial H}{\partial Y_1} \\ & + [g_{Y_2} (Y_{1t}, Y_{2t}) - \phi_2] \frac{\partial H}{\partial Y_2} - r_t H + \Lambda (W_t, Y_{1t}, Y_{2t}, t) , \end{aligned} \tag{17}$$

where (i) ϕ_1, ϕ_2 and r_t are given by (14), (15) and (16), respectively and (ii) $\Lambda(W_t, Y_{1t}, Y_{2t}, t)$ represents the instantaneous payoff of the contingent claim, which depends on its specific contractual conditions.

From the PDE (17), it is clear that ambiguity aversion impacts the fundamental pricing equation through two preference-dependent inputs: the equilibrium instantaneous interest rate and the equilibrium price of uncertainty associated with the ambiguous state variable Y_{2t}.

A particular case of this setting is presented in Longstaff and Schwartz (1992), where an intertemporal general equilibrium setting for valuing interest rate sensitive contingent claims is developed starting from a two state variable version of the model of Cox et al. (1985a). There, it is also assumed that

[9] Regarding (16), depending on the specifications of g_Q and σ_Q, conditions on parameters have to be imposed in order to guarantee that the equilibrium interest rate is non-negative. As explained by Longstaff and Schwartz (1992), the lower bound of zero for the interest rate is consistent with the basic properties of the economy under study, because as the single good produced in this economy can be consumed or invested in the production process, it can be seen as storable.

[10] As highlighted in Epstein and Schneider (2010), agent's willingness to save is a positive function of his level of uncertainty and the more the agent tries to save the lower tends to be the equilibrium interest rate. Considering ambiguity as an extra source of uncertainty (alongside risk), we may therefore conclude that the most intuitive scenario is that when ambiguity is considered, the equilibrium interest rate decreases (everything else constant). This is also the result under the general equilibrium model with ambiguity in Trojani and Vanini (2004).

both state variables impact the expected output rate in the economy, that only one of them has shocks correlated with those of the economy's output rate, and that the representative agent has a logarithmic utility function. They assume, however, that the state variables are uncorrelated. Moreover, in their setting, uncertainty is exclusively risk, i.e., there is no ambiguity. The model of Longstaff and Schwartz (1992) is, therefore, a particular case of our setting (see Appendix 5.3).

3 An Example of Investment Opportunity Set

In the previous section, we developed, for a general investment opportunity set with endogenous production driven by two correlated state variables, an intertemporal equilibrium setting for contingent claim pricing following Cox et al. (1985a) and Gagliardini et al. (2009), considering that uncertainty includes two dimensions: risk and ambiguity. A key characteristic of our investment opportunity set is that the state variables, Y_{1t} and Y_{2t}, are correlated but only one of them, Y_{2t}, has shocks correlated with those of the economy's output rate, and, simultaneously, the representative agent is ambiguous about its stochastic process.

In this section, as an example, we apply results of the previous section by considering a concrete investment opportunity set, where the state variable Y_{1t} is an economic variable and Y_{2t} represents the variance of changes in Y_{1t}.

Additionally, the return of economy's production process (output rate) is assumed to be given by:

$$\frac{dQ_t}{Q_t} = g_Q(Y_{1t}, Y_{2t})dt + l\sqrt{Y_{2t}}dW_Q\,, \tag{18}$$

where the drift function, g_Q, is, for now, still unspecified and $l > 0$.

It is assumed that Y_{1t} follows the geometric Brownian motion:

$$dY_{1t} = \mu Y_{1t}dt + Y_{1t}\sqrt{Y_{2t}}dW_1, \tag{19}$$

where μ is the expected growth rate of Y_{1t}, Y_{2t} is its instantaneous variance and it is assumed that $dW_Q dW_1 = 0$.

Regarding the ambiguous state variable Y_{2t}, the "reference belief" dynamics for the representative agent is assumed to be given by the mean reverting square-root process (as used, for example, in Cox et al. (1985b)):

$$dY_{2t} = \kappa\left(\theta - Y_{2t}\right)dt + \epsilon\sqrt{Y_{2t}}dW_2, \tag{20}$$

where θ is the expected value of Y_{2t}, $\kappa > 0$ is the mean reverting parameter and $\epsilon > 0$. It is assumed that $dW_1 dW_2 = \rho dt$ and $dW_Q dW_2 = \rho_2 dt$.

From (18), (19) and (20), we have $\sigma_Q = l\sqrt{Y_{2t}}$, $\sigma_{Y_1} = Y_{1t}\sqrt{Y_{2t}}$ and $\sigma_{Y_2} = \epsilon\sqrt{Y_{2t}}$, therefore, from (15), it is straightforward to obtain the specification of ϕ_2, the equilibrium market price of uncertainty associated with Y_{2t}:

$$\phi_2 = \underbrace{\epsilon l \rho_2 Y_{2t}}_{risk\,price} \underbrace{\pm\epsilon\sqrt{2\eta\left(1-\rho^2\right)Y_{2t}}}_{ambiguity\,price},$$

which, by defining $\lambda_1 = \epsilon l \rho_2$ and $\lambda_2 = \pm\epsilon\sqrt{2\eta\left(1-\rho^2\right)}$, can be written as:

$$\phi_2 = \lambda_1 Y_{2t} + \lambda_2\sqrt{Y_{2t}}. \tag{21}$$

The equilibrium market price of uncertainty associated with the variance of changes in Y_{1t} has two components: the variance risk price, which is linear on the instantaneous level of variance, Y_{2t}, and the variance ambiguity price, which is proportional to the square-root of Y_{2t}. The variance risk price depends on: (i) the parameter l of the diffusion function in the stochastic process describing the economy's output rate; (ii) the parameter ϵ of the diffusion function of the stochastic process of Y_{2t}; and (iii) on the correlation ρ_2 between shocks in dY_{2t} and in the output rate. The variance risk price is positive (negative) when $\rho_2 > 0$ ($\rho_2 < 0$), since, by assumption, $\epsilon > 0$, $l > 0$ (see also footnote 5). The variance ambiguity price depends on ϵ, on the correlation of shocks in both state variables (ρ), and on the degree of ambiguity faced by the representative agent (η). It can also be positive or negative.

Note that the specification for the dynamics of the state variables (19) and (20) is the one that is used in Heston's (1993) stochastic volatility option pricing model. In Heston (1993), Y_{1t} represents the option's underlying asset spot price and, consequently, Y_{2t} is the variance of the underlying asset return, with both being correlated. The specification for the market price of variance risk used in Heston (1993), where uncertainty is exclusively risk, is a scalar multiplied by the instantaneous level of variance. This is consistent with our findings under the developed equilibrium approach: it corresponds to the $\lambda_1 Y_{2t}$ component in (21). We have therefore provided an equilibrium motivation for the price specification of variance risk in Heston's (1993) model. We also conclude that a potential extension of Heston's (1993) model by incorporating ambiguity aversion about the stochastic variance process of the underlying asset return could use (21) as the specification for the variance uncertainty price, with the sign of the ambiguity component depending on the concrete calibration to be used.

Moreover, from (16), it is straightforward to obtain the expression for the equilibrium interest rate:

$$r_t \quad = \quad g_Q - l^2 Y_{2t} \pm \frac{l\rho_2}{\sqrt{1-\rho^2}} \sqrt{2\eta}\sqrt{Y_{2t}}\,, \tag{22}$$

where the new component emerging from the ambiguity consideration is $\pm\frac{l\rho_2}{\sqrt{1-\rho^2}}\sqrt{2\eta}\sqrt{Y_{2t}}$.

In order to study the sign of the ambiguity components in both (21) and (22), we must specify the output rate drift function g_Q and, subsequently, solve the corresponding Bellman equation (12). This is illustrated in the next subsection.

3.1 A Particular Solution

Assuming $g_Q(Y_{1t}, Y_{2t}) = lnY_{1t} + \alpha Y_{2t}$, where α is a scalar parameter, the output rate process (18) is given by:

$$\frac{dQ_t}{Q_t} = (lnY_{1t} + \alpha Y_{2t})dt + l\sqrt{Y_{2t}}dW_Q\,. \tag{23}$$

Considering this concrete specification for the output rate process and the processes (19) and (20) for the state variables, we start by solving the corresponding Bellman equation (12), which is given by (Appendix 5.4.1):

$$\begin{aligned}\delta V \quad = \quad & \mu V_{Y_1} Y_{1t} + \kappa\left(\theta - Y_{2t}\right) V_{Y_2} + lnY_{1t} + \alpha Y_{2t} - \frac{1}{2} l^2 Y_{2t} \\ & + \frac{1}{2}\left(Y_{1t}^2 Y_{2t} V_{Y_1Y_1} + 2\epsilon\rho V_{Y_2Y_1} Y_{1t} Y_{2t} + \epsilon^2 Y_{2t} V_{Y_2Y_2}\right) - \sqrt{2\eta F\left(Y_{1t}, Y_{2t}\right)}\,,\end{aligned} \tag{24}$$

where,

$$\begin{aligned}F\left(Y_{1t}, Y_{2t}\right) \quad = \quad & l^2 Y_{2t}\left(1 - \frac{\rho_2^2}{1-\rho^2}\right) + \left(V_{Y_1} Y_{1t}\sqrt{Y_{2t}} + \epsilon\sqrt{Y_{2t}}\rho V_{Y_2}\right)^2 \\ & + \left(\epsilon\sqrt{Y_{2t}}\sqrt{1-\rho^2} V_{Y_2} + \frac{l\sqrt{Y_{2t}}\rho_2}{\sqrt{1-\rho^2}}\right)^2.\end{aligned}$$

We obtain a solution that is exact when there is no ambiguity ($\eta = 0$) and approximate in the presence of ambiguity (Appendix 5.4.2):

$$
\begin{aligned}
V\left(Y_{1t}, Y_{2t}\right) &= alnY_{1t} + bY_{2t} + c\,, \qquad (25)\\
&\text{with}\\
a &= \frac{1}{\delta}\,,\\
b &= \frac{\left(\alpha - \frac{l^2}{2} - \frac{1}{2\delta}\right)}{(\kappa+\delta)}\,,\\
c &= \frac{\mu}{\delta^2} + \frac{\kappa\theta}{\delta}b\,.
\end{aligned}
$$

The value function (25) is an approximate solution of (24) in the domain $0 < \eta < \Psi$ (with Ψ being an arbitrarily small positive number), assuming that V_{Y_2} and $\frac{\partial V_{Y_2}}{\partial \eta}$ exist. It is difficult to obtain an exact solution for (24) under ambiguity ($\eta > 0$). We suspect that, if a solution exists, it is not separable in the state variables, making it difficult to study its gradient with respect to Y_{2t}, and numerical procedures are necessary to find it. Moreover, there is a reason to believe that the accuracy of the approximation is reasonable: the domain $0 < \eta < \Psi$ must be very tight, for the reasoning previously invoked that alternative models must be statistically close to the "reference belief" model, so that the representative agent has difficulty to distinguish them and therefore faces ambiguity.

This asymptotic method of finding an approximate solution of the problem is intuitively close to the perturbation theory used in Trojani and Vanini (2004) to solve intertemporal general equilibrium models under ambiguity. The rationale is provided by the authors (p. 291) "*the basic idea of asymptotic methods is to formulate a general problem, find a particular relevant case that has a known solution, and use this as a starting point for computing the solution to nearby problems*". As in our case, in Trojani and Vanini (2004), the asymptotic solutions of the problems under ambiguity "... *hold for neighborhoods of a model with log utility of consumption and no ambiguity aversion*".

From (25), it is immediate that $V_{Y_2} = b$, and the expression for the equilibrium price of uncertainty associated with Y_2 (21) can be clarified (Appendix 5.4.3):

$$
\phi_2 \;=\; \lambda_1 Y_{2t} + \lambda_2\sqrt{Y_{2t}}\;, \text{ with } \lambda_2 > 0 \;\; (<0) \;\text{ if }\; \alpha > \omega \;\; (<\omega)\,, \qquad (26)
$$

and the threshold value ω being given by:

$$
\omega \;=\; \frac{\left(l^2\delta\epsilon + \epsilon\right)\left(1-\rho^2\right) - 2\delta\left(\kappa+\delta\right) l\rho_2}{2\delta\epsilon\left(1-\rho^2\right)}\,. \qquad (27)
$$

The equilibrium uncertainty price associated with Y_{2t} can therefore increase or decrease when ambiguity about its process is considered. That depends on the relative magnitude of the parameter α, which measures the sensitivity of the expected output rate of the economy relatively to changes on Y_{2t}, versus a benchmark value that synthesizes some information of the investment opportunity set (parameters l, ϵ, κ, ρ and ρ_2) and the subjective rate of discount of the representative agent, δ. If $\alpha > \omega$, ambiguity about the stochastic process of Y_{2t} increases its equilibrium uncertainty price, and the contrary when $\alpha < \omega$.

Regarding the expression for the equilibrium instantaneous interest rate, r_t, under this concrete setting, from (16) it is given by (Appendix 5.4.3):

$$
r_t \;=\; \begin{cases} \left(lnY_{1t} + \alpha Y_{2t}\right) - l^2 Y_{2t} - \frac{l\rho_2}{\sqrt{1-\rho^2}}\sqrt{2\eta}\sqrt{Y_{2t}} & \text{if } \alpha > \omega\,,\\[2ex] \left(lnY_{1t} + \alpha Y_{2t}\right) - l^2 Y_{2t} + \frac{l\rho_2}{\sqrt{1-\rho^2}}\sqrt{2\eta}\sqrt{Y_{2t}} & \text{if } \alpha < \omega\,. \end{cases} \qquad (28)
$$

From (28), one concludes that if there exists a negative correlation between shocks in the ambiguous state variable and the economy's output rate ($\rho_2 < 0$), the impact on the equilibrium interest rate from ambiguity has the same direction (increase or decrease) as on the equilibrium uncertainty price. The contrary happens when $\rho_2 > 0$.

Overall, the effects on ϕ_2 and r_t from the consideration of ambiguity about the stochastic process of Y_{2t} are summarized in Table 1:

Table 1: Impact on ϕ_2 and r_t from ambiguity about the stochastic process of Y_{2t}

	$\alpha > \omega$		$\alpha < \omega$	
	$\rho_2 < 0$	$\rho_2 > 0$	$\rho_2 < 0$	$\rho_2 > 0$
ϕ_2	↑	↑	↓	↓
r_t	↑	↓	↓	↑

Note: The sign ↑ (↓) indicates that ambiguity about Y_{2t} stochastic process increases (decreases) ϕ_2 and r_t.

4 Concluding Remarks

We developed a general intertemporal equilibrium setting for asset pricing using a two state variable version of the model of Cox et al. (1985a). All the physical investment is delivered by a single stochastic production process whose realized return (economy's output rate) is driven by two state variables, Y_{1t} and Y_{2t}. It is assumed that both state variables impact the economy's expected output rate, but only one of them (Y_{2t}) has shocks correlated with those of the output rate. A key assumption in our setting is that the state variables are correlated, which we believe to be quite useful for modeling economic problems, particularly regarding asset pricing.

It is assumed that the representative agent, with a logarithmic utility function, is not totally sure about the probability measure P under which his investment opportunity set evolves. More, precisely, it is assumed that the representative agent is ambiguous about the stochastic model that characterizes the dynamics of the state variable Y_{2t}. The representative agent considers contaminations around his reference belief and aversion towards ambiguity is introduced by assuming that, in the spirit of Gilboa and Schmeidler (1989), the representative agent chooses from all the contaminations the one associated with lower expected utility. Ambiguity aversion changes the fundamental pricing equation satisfied by the contingent claim price through two inputs, that become preference-dependent: the equilibrium instantaneous interest rate and the equilibrium price of uncertainty associated with Y_{2t}. Those two inputs embed two components, corresponding to the two uncertainty dimensions: risk and ambiguity.

It is found that the equilibrium market price of risk associated with the state variable Y_{2t} depends on its correlation with the economy's output rate shocks and on the diffusion functions of the stochastic processes of the output rate and of Y_{2t}. The equilibrium interest rate, when uncertainty is exclusively risk, is found to be given by the difference between the expected output rate of the economy and the variance of the output rate. Longstaff and Schwartz's (1992) general equilibrium model for the interest rate term structure can be obtained as a particular case of our setting.

When ambiguity is considered, the equilibrium price of uncertainty associated with Y_{2t} and the equilibrium interest rate can increase or decrease, depending on (i) the impact of changes in the ambiguous state variable, Y_{2t}, on the indirect utility (value function); (ii) the correlations between shocks in Y_{2t} and shocks in the other state variable, Y_{1t}, and in the economy's output rate; and (iii) the diffusion functions of the stochastic processes for the ambiguous state variable, Y_{2t}, and the economy's output rate.

As an example, we apply the obtained general results to a specific investment opportunity set, where Y_{2t} is the instantaneous variance of the change of Y_{1t}, continuing to assume that both are correlated but only Y_{2t} has shocks correlated with those of economy's output rate. This contains the investment opportunity set of the well known option pricing model of Heston (1993), by letting Y_{1t} be the option's underlying asset price. The obtained equilibrium market price of variance risk is linear on its instantaneous level, which in fact is the specification used by Heston (1993): we therefore provide an equilibrium motivation for the specification of the price of variance risk used by Heston (1993). The obtained equilibrium market price of ambiguity about stochastic variance is proportional to the square-root of its instantaneous level. Through this example, we have therefore obtained a specification of the equilibrium variance uncertainty price that can be used in an extension of Heston (1993) model that accommodates ambiguity aversion about the stochastic variance process of the option's underlying asset return. This extension is carried out in Faria and Correia-da Silva (2010).

5 Appendix

5.1 Correlation Structure

In the setting of Cox et al. (1985a) the Brownian motions that impact the dynamics of the output rate and the state variables are assumed to be independent. It is possible to rewrite the diffusion component of the system (1), (2) and (3) in a way that, maintaining the desired correlation structure, there is a vector of independent Brownian Motions and, consequently, making it possible to apply the results of Cox et al. (1985a).

The diffusion component of the system (1), (2) and (3) is given by:

$$\begin{bmatrix} \sigma_Q & 0 & 0 \\ 0 & \sigma_{Y_1} & 0 \\ 0 & 0 & \sigma_{Y_2} \end{bmatrix} \begin{bmatrix} dW_Q \\ dW_1 \\ dW_2 \end{bmatrix},$$

where (i) $dW_Q dW_1 = 0$, $dW_Q dW_2 = \rho_2 dt$ and $dW_1 dW_2 = \rho dt$, (ii) the variance of the output rate, of dY_{1t} and of dY_{2t} is given by σ_Q^2, $\sigma_{Y_1}^2$ and $\sigma_{Y_2}^2$, respectively and, consequently, (iii) $Cov\left(\frac{dQ_t}{Q_t},\, Y_{1t}\right) = 0$, $Cov\left(\frac{dQ_t}{Q_t},\, Y_{2t}\right) = \sigma_Q \sigma_{Y_{2t}} \rho_2$ and $Cov\left(Y_{1t},\, Y_{2t}\right) = \sigma_{Y_{1t}} \sigma_{Y_{2t}} \rho$, where $Cov\left(\cdot\right)$ stands for the covariance.

In order to maintain this correlation structure when considering the vector Z of independent Brownian Motions in (5), it is necessary that:

$$\begin{bmatrix} \sigma_Q & 0 & 0 \\ 0 & \sigma_{Y_1} & 0 \\ 0 & 0 & \sigma_{Y_2} \end{bmatrix} \begin{bmatrix} dW_Q \\ dW_1 \\ dW_2 \end{bmatrix} = A \begin{bmatrix} dZ_0 \\ dZ_1 \\ dZ_2 \end{bmatrix},$$

with the generic matrix A satisfying the conditions

$$\begin{aligned} a_{11}^2 + a_{12}^2 + a_{13}^2 &= \sigma_Q^2, \\ a_{21}^2 + a_{22}^2 + a_{23}^2 &= \sigma_{Y_1}^2, \\ a_{31}^2 + a_{32}^2 + a_{33}^2 &= \sigma_{Y_2}^2, \\ a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} &= 0, \\ a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} &= \sigma_Q \sigma_{Y_2} \rho_2, \\ a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} &= \sigma_{Y_1} \sigma_{Y_2} \rho, \end{aligned} \tag{29}$$

where a_{ij} represents the element in the i^{th} line and j^{th} column of matrix A.

It is immediate to conclude that the matrix A in (4) satisfies the conditions (29), as:

$$\begin{aligned}
\sigma_Q^2\left(1-\frac{\rho_2^2}{1-\rho^2}\right)+\frac{\sigma_Q^2\rho_2^2}{1-\rho^2} &= \sigma_Q^2\,,\\
\sigma_{Y_1}^2 &= \sigma_{Y_1}^2,\\
\sigma_{Y_2}^2\rho^2+\sigma_{Y_2}^2\left(1-\rho^2\right) &= \sigma_{Y_2}^2,\\
0 &= 0,\\
\frac{\sigma_Q\rho_2}{\sqrt{1-\rho^2}}\sigma_{Y_2}\sqrt{1-\rho^2} &= \sigma_Q\sigma_{Y_2}\rho_2,\\
\sigma_{Y_1}\sigma_{Y_2}\rho &= \sigma_{Y_1}\sigma_{Y_2}\rho.
\end{aligned}$$

□

5.2 Expressions (13)-(16)

5.2.1 Optimal Contamination Drift (13)

In order to obtain the equilibrium contamination drift vector $h^* = \begin{bmatrix} 0 & 0 & h_2 \end{bmatrix}^\top$, we make use of Proposition 1 in Gagliardini et al. (2009), which implies that:

$$h_2 = -\sqrt{2\eta}\frac{\sigma_{Y_2}\sqrt{1-\rho^2}V_{Y_2}+\frac{\sigma_Q\rho_2}{\sqrt{1-\rho^2}}}{\sqrt{\left(\sigma_{Y_2}\sqrt{1-\rho^2}V_{Y_2}+\frac{\sigma_Q\rho_2}{\sqrt{1-\rho^2}}\right)^2}}\,, \tag{30}$$

from which it is immediate to obtain (13).

□

5.2.2 Equilibrium Price of Uncertainty (14)-(15)

From Corollary 1 in Gagliardini et al. (2009) the equilibrium market premium of risk and ambiguity (M) associated with the state variables Y_{1t} and Y_{2t} and the production process Q_t is given by:

$$M = \sigma^\top - h^*,$$

with σ given by (5) and the equilibrium contamination drift vector $h^* = \begin{bmatrix} 0 & 0 & h_2 \end{bmatrix}^\top$given by (13). It is immediate to conclude that M is given by:

$$M = \begin{bmatrix} \sigma_Q\sqrt{\left(1-\frac{\rho_2^2}{1-\rho^2}\right)} \\ 0 \\ \frac{\sigma_Q\rho_2}{\sqrt{1-\rho^2}} \pm\sqrt{2\eta} \end{bmatrix}. \tag{31}$$

Following expression [18] in Gagliardini et al. (2009), the equilibrium market prices of uncertainty

ϕ_1 and ϕ_2 associated with Y_{1t} and Y_{2t}, respectively, are given by:

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \Xi M = \begin{bmatrix} 0 & \sigma_{Y_1} & 0 \\ 0 & \sigma_{Y_2}\rho & \sigma_{Y_2}\sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} \sigma_Q\sqrt{\left(1-\frac{\rho_2^2}{1-\rho^2}\right)} \\ 0 \\ \frac{\sigma_Q\rho_2}{\sqrt{1-\rho^2}} \pm \sqrt{2\eta} \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \sigma_Q\sigma_{Y_2}\rho_2 \pm \sigma_{Y_2}\sqrt{2\eta}\sqrt{1-\rho^2} \end{bmatrix},$$

which, from (30), can be written as:

$$\phi_1 = 0,$$

$$\phi_2 = \begin{cases} \sigma_{Y_2}\sigma_Q\rho_2 + \sigma_{Y_2}\sqrt{2\eta}\sqrt{1-\rho^2}, \text{ if } \sigma_{Y_2}\sqrt{1-\rho^2}V_{Y_2} + \frac{\sigma_Q\rho_2}{\sqrt{1-\rho^2}} > 0, \\ \sigma_{Y_2}\sigma_Q\rho_2 - \sigma_{Y_2}\sqrt{2\eta}\sqrt{1-\rho^2}, \text{ if } \sigma_{Y_2}\sqrt{1-\rho^2}V_{Y_2} + \frac{\sigma_Q\rho_2}{\sqrt{1-\rho^2}} < 0, \end{cases}$$

which is (14) and (15), respectively.

□

5.2.3 Equilibrium Interest Rate (16)

From Corollary 1 in Gagliardini et al. (2009) the equilibrium instantaneous interest rate r_t is given by:

$$r_t = g_Q - \sigma M,$$

where g_Q is the drift function in (1) and matrices σ and M are those given in (5) and (31), respectively. Consequently, r_t is given by:

$$r_t = g_Q - \begin{bmatrix} \sigma_Q\sqrt{\left(1-\frac{\rho_2^2}{1-\rho^2}\right)} & 0 & \frac{\sigma_Q\rho_2}{\sqrt{1-\rho^2}} \end{bmatrix} \begin{bmatrix} \sigma_Q\sqrt{\left(1-\frac{\rho_2^2}{1-\rho^2}\right)} \\ 0 \\ \frac{\sigma_Q\rho_2}{\sqrt{1-\rho^2}} \pm \sqrt{2\eta} \end{bmatrix}$$

$$\Leftrightarrow r_t = g_Q - \sigma_Q^2 \pm \sqrt{2\eta}\frac{\sigma_Q\rho_2}{\sqrt{1-\rho^2}},$$

which from the previous section 5.2.2, can be written as:

$$r_t = \begin{cases} g_Q - \sigma_Q^2 - \sqrt{2\eta}\frac{\sigma_Q\rho_2}{\sqrt{1-\rho^2}}, \text{ if } \sigma_{Y_2}\sqrt{1-\rho^2}V_{Y_2} + \frac{\sigma_Q\rho_2}{\sqrt{1-\rho^2}} > 0, \\ g_Q - \sigma_Q^2 + \sqrt{2\eta}\frac{\sigma_Q\rho_2}{\sqrt{1-\rho^2}}, \text{ if } \sigma_{Y_2}\sqrt{1-\rho^2}V_{Y_2} + \frac{\sigma_Q\rho_2}{\sqrt{1-\rho^2}} < 0, \end{cases}$$

which is (16).

□

5.3 Longstaff and Schwartz (1992): a Particular Case

The model of Longstaff and Schwartz (1992) can be obtained as a particular case of our setting. Start by assuming:

$$g_Q = (zY_{1t} + uY_{2t}) \text{ and } \sigma_Q = \nu\sqrt{Y_{2t}},$$

$$g_{Y_1} = (a - bY_{1t}) \text{ and } \sigma_{Y_1} = c\sqrt{Y_{1t}},$$

$$g_{Y_2} = (d - eY_{2t}) \text{ and } \sigma_{Y_2} = f\sqrt{Y_{2t}}\,,$$

where z, u, ν, a, b, c, d, e and f are positive parameters.

And make the following change of variables, $y_{1t} = \frac{Y_{1t}}{c^2}$ and $y_{2t} = \frac{Y_{2t}}{f^2}$, which implies:

$$g_{y_1} = (\gamma - \vartheta y_{1t}) \text{ and } \sigma_{y_1} = \sqrt{y_{1t}},$$

$$g_{y_2} = (\zeta - \xi y_{2t}) \text{ and } \sigma_{y_2} = \sqrt{y_{2t}},$$

where $\gamma = \frac{a}{c^2}$, $\vartheta = b$, $\zeta = \frac{d}{f^2}$ and $\xi = e$.

In the model of Longstaff and Schwartz (1992), the state variable Y_{1t} has uncorrelated shocks with those of economy's output rate and of the other state variable Y_{2t}, i.e., $\rho = 0$. Thus, matrix A in (4) becomes:

$$\begin{bmatrix} \sigma_Q\sqrt{(1-\rho_2^2)} & 0 & \sigma_Q\rho_2 \\ 0 & \sigma_{Y_1} & 0 \\ 0 & 0 & \sigma_{Y_2} \end{bmatrix}.$$

Considering those specifications and the assumption in Longstaff and Schwartz (1992) that $\Lambda\left(W_t, Y_{1t}, Y_{2t}, t\right) = 0$, from (17) one obtains the fundamental PDE satisfied by the contingent claim price $H\left(y_{1t}, y_{2t}, \tau\right)$:

$$\begin{aligned} \frac{\partial H}{\partial \tau} &= \frac{1}{2}\left(\sqrt{y_{1t}}\right)^2 \frac{\partial^2 H}{\partial y_1^2} + \frac{1}{2}\left(\sqrt{y_{2t}}\right)^2 \frac{\partial^2 H}{\partial y_2^2} + (\gamma - \vartheta y_{1t})\frac{\partial H}{\partial y_1} + \\ &\quad + \left[(\zeta - \xi y_{2t}) - \sqrt{y_{2t}}\nu f\sqrt{y_{2t}}\rho_2\right]\frac{\partial H}{\partial y_2} - r_t H \\ \Leftrightarrow \frac{\partial H}{\partial \tau} &= \frac{y_{1t}}{2}\frac{\partial^2 H}{\partial y_1^2} + \frac{y_{2t}}{2}\frac{\partial^2 H}{\partial y_2^2} + (\gamma - \vartheta y_{1t})\frac{\partial H}{\partial y_1} + \\ &\quad + \left[(\zeta - \xi y_{2t}) - \lambda y_{2t}\right]\frac{\partial H}{\partial y_2} - r_t H\,, \end{aligned}$$

with $\lambda = \rho_2 f \nu$, which is the PDE obtained by Longstaff and Schwartz (1992) (their equations (8) and (9)).[11]

Regarding the equilibrium interest rate, from (16) and the above specifications:

$$r_t = g_Q - \sigma_Q^2 = zc^2 y_{1t} + f^2\left(u - \nu^2\right) y_{2t}\,,$$

which is the expression reached by Longstaff and Schwartz (1992) (their equation (10)).

□

[11] Note that the equilibrium price of risk associated with the "original" state variable Y_{2t} is also linear on its instantaneous level with the same coefficient λ, as $\sigma_Q\sigma_{Y_2}\rho_2 = \nu\sqrt{Y_{2t}}f\sqrt{Y_{2t}}\rho_2 = \rho_2\nu f Y_{2t} = \lambda Y_{2t}$, with $\lambda = \rho_2 f\nu$.

5.4 Concrete Investment Opportunity Set

5.4.1 Bellman Equation (24)

Under the investment opportunity set (19), (20) and (23):

$$\begin{aligned} g_Y &= \begin{bmatrix} \mu Y_{1t} \\ \kappa(\theta - Y_{2t}) \end{bmatrix}; \\ \Xi &= \begin{bmatrix} 0 & Y_{1t}\sqrt{Y_{2t}} & 0 \\ 0 & \epsilon\sqrt{Y_{2t}}\rho & \epsilon\sqrt{Y_{2t}}\sqrt{1-\rho^2} \end{bmatrix}; \\ \sigma^\top &= \begin{bmatrix} l\sqrt{Y_{2t}}\sqrt{1-\frac{\rho_2^2}{1-\rho^2}} \\ 0 \\ \frac{l\sqrt{Y_{2t}}\rho_2}{\sqrt{1-\rho^2}} \end{bmatrix}. \end{aligned}$$

From which, it is immediate to obtain:

$$V_Y^\top g_Y = \begin{bmatrix} V_{Y_1} & V_{Y_2} \end{bmatrix} \begin{bmatrix} \mu Y_{1t} \\ \kappa(\theta - Y_{2t}) \end{bmatrix} = \mu V_{Y_1} Y_{1t} + \kappa(\theta - Y_{2t}) V_{Y_2};$$

$$\begin{aligned} \Xi^\top V_{YY} \Xi &= \begin{bmatrix} 0 & 0 \\ Y_{1t}\sqrt{Y_{2t}} & \epsilon\sqrt{Y_{2t}}\rho \\ 0 & \epsilon\sqrt{Y_{2t}}\sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} V_{Y_1Y_1} & V_{Y_1Y_2} \\ V_{Y_2Y_1} & V_{Y_2Y_2} \end{bmatrix} \begin{bmatrix} 0 & Y_{1t}\sqrt{Y_{2t}} & 0 \\ 0 & \epsilon\sqrt{Y_{2t}}\rho & \epsilon\sqrt{Y_{2t}}\sqrt{1-\rho^2} \end{bmatrix} \\ \Rightarrow trace\left[\Xi^\top V_{YY}\Xi\right] &= Y_{1t}^2 Y_{2t} V_{Y_1Y_1} + 2\epsilon\rho V_{Y_2Y_1} Y_{1t} Y_{2t} + \epsilon^2 Y_{2t} V_{Y_2Y_2}; \end{aligned}$$

$$\begin{aligned} \Xi^\top V_Y + \sigma^\top &= \begin{bmatrix} 0 & 0 \\ Y_{1t}\sqrt{Y_{2t}} & \epsilon\sqrt{Y_{2t}}\rho \\ 0 & \epsilon\sqrt{Y_{2t}}\sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} V_{Y_1} \\ V_{Y_2} \end{bmatrix} + \begin{bmatrix} l\sqrt{Y_{2t}}\sqrt{1-\frac{\rho_2^2}{1-\rho^2}} \\ 0 \\ \frac{l\sqrt{Y_{2t}}\rho_2}{\sqrt{1-\rho^2}} \end{bmatrix} \\ \Leftrightarrow \Xi^\top V_Y + \sigma^\top &= \begin{bmatrix} l\sqrt{Y_{2t}}\sqrt{1-\frac{\rho_2^2}{1-\rho^2}} \\ V_{Y_1} Y_{1t}\sqrt{Y_{2t}} + \epsilon\sqrt{Y_{2t}}\rho V_{Y_2} \\ \epsilon\sqrt{Y_{2t}}\sqrt{1-\rho^2} V_{Y_2} + \frac{l\sqrt{Y_{2t}}\rho_2}{\sqrt{1-\rho^2}} \end{bmatrix} \\ \Rightarrow \left(\Xi^\top V_Y + \sigma^\top\right)^\top \left(\Xi^\top V_Y + \sigma^\top\right) &= l^2 Y_{2t}\left(1 - \frac{\rho_2^2}{1-\rho^2}\right) + \left(V_{Y_1} Y_{1t}\sqrt{Y_{2t}} + \epsilon\sqrt{Y_{2t}}\rho V_{Y_2}\right)^2 \\ &\quad + \left(\epsilon\sqrt{Y_{2t}}\sqrt{1-\rho^2} V_{Y_2} + \frac{l\sqrt{Y_{2t}}\rho_2}{\sqrt{1-\rho^2}}\right)^2 = F(Y_{1t}, Y_{2t}). \end{aligned}$$

Going back to (12), and substituting the obtained expressions:

$$\begin{aligned} \delta V &= \mu V_{Y_1} Y_{1t} + \kappa(\theta - Y_{2t}) V_{Y_2} + lnY_{1t} + \alpha Y_{2t} - \frac{1}{2} l^2 Y_{2t} \\ &\quad + \frac{1}{2}\left(Y_{1t}^2 Y_{2t} V_{Y_1Y_1} + 2\epsilon\rho V_{Y_2Y_1} Y_{1t} Y_{2t} + \epsilon^2 Y_{2t} V_{Y_2Y_2}\right) \\ &\quad - \sqrt{2\eta F(Y_{1t}, Y_{2t})}, \end{aligned}$$

which is (24).

□

5.4.2 Value Function (25)

Start by noting that the value function $V(Y_{1t}, Y_{2t})$ is a function of the two state variables, Y_{1t} and Y_{2t}, for a given ambiguity parameter η, and therefore can be denoted as $V(Y_{1t}, Y_{2t}, \eta)$:

$$V:\ \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}\,.$$

When there is no ambiguity ($\eta = 0$), the Bellman equation (24) becomes:

$$\begin{aligned} \delta V &= \mu V_{Y_1} Y_{1t} + \kappa(\theta - Y_{2t}) V_{Y_2} + \frac{1}{2}\left(Y_{1t}^2 Y_{2t} V_{Y_1Y_1} + 2\epsilon\rho V_{Y_2Y_1} Y_{1t} Y_{2t} + \epsilon^2 Y_{2t} V_{Y_2Y_2}\right) \\ &\quad + lnY_{1t} + \alpha Y_{2t} - \frac{1}{2} l^2 Y_{2t}\,, \end{aligned} \tag{32}$$

which is solved by the value function:

$$V(Y_{1t}, Y_{2t}, \eta = 0) = alnY_{1t} + bY_{2t} + c\,, \tag{33}$$

where:

$$\begin{aligned} a &= \frac{1}{\delta}\,, \\ b &= \frac{\left(\alpha - \frac{l^2}{2} - \frac{1}{2\delta}\right)}{(\kappa + \delta)}\,, \\ c &= \frac{\mu}{\delta^2} + \frac{\kappa\theta}{\delta} b\,. \end{aligned} \tag{34}$$

Proof:

From (33):

$$\begin{aligned} V_{Y_1} &= \frac{a}{Y_{1t}}\,,\ V_{Y_1Y_1} = -\frac{a}{Y_{1t}^2}\,,\ V_{Y_1Y_2} = 0\,, \\ V_{Y_2} &= b\,,\ V_{Y_2Y_2} = 0\,. \end{aligned}$$

Considering those results, (33) and (34), and plugging them into (32) one gets:

$$\begin{aligned} \delta\left(\frac{1}{\delta} lnY_{1t} + bY_{2t} + c\right) &= \mu\frac{a}{Y_{1t}} Y_{1t} + \kappa(\theta - Y_{2t}) b + \frac{1}{2}\left(-Y_{1t}^2 Y_{2t} \frac{a}{Y_{1t}^2}\right) + lnY_{1t} \\ &\quad + \alpha Y_{2t} - \frac{l^2}{2} Y_{2t} \\ \Leftrightarrow lnY_{1t} + \delta bY_{2t} + \delta c &= \mu a + \kappa(\theta - Y_{2t}) b - \frac{1}{2} Y_{2t} a + lnY_{1t} + \alpha Y_{2t} - \frac{l^2}{2} Y_{2t} \\ \Leftrightarrow \delta c - \frac{\mu}{\delta} - b\kappa\theta &= \left[\alpha - \frac{l^2}{2} - \frac{1}{2\delta} - b(\kappa + \delta)\right] Y_{2t} \\ \Leftrightarrow \frac{\mu}{\delta} + \kappa\theta b - \frac{\mu}{\delta} - b\kappa\theta &= \left[\alpha - \frac{l^2}{2} - \frac{1}{2\delta} - \frac{\left(\alpha - \frac{l^2}{2} - \frac{1}{2\delta}\right)}{(\kappa + \delta)}(\kappa + \delta)\right] Y_{2t} \\ \Leftrightarrow 0 &= 0, \end{aligned}$$

as we wanted to prove.

From previous assumptions that V_{Y_2} and $\frac{\partial V_{Y_2}}{\partial \eta}$ exist in the domain $0 < \eta < \Psi$ (with Ψ being a small positive number), results that in this domain for η, $V(Y_{1t}, Y_{2t}, \eta > 0) \approx V(Y_{1t}, Y_{2t}, \eta = 0)$, implying $V_{Y_2}|_{\eta>0} \approx V_{Y_2}|_{\eta=0} = b$, where b is given by (34).

□

5.4.3 Equilibrium Uncertainty Price (26) and Interest Rate (28)

From (15) and (16), a key issue is the sign of $V_{Y_2}\sigma_{Y_2}(1-\rho^2) + \sigma_Q\rho_2$. Under the concrete investment opportunity set:

$$V_{Y_2}\sigma_{Y_2}\left(1-\rho^2\right) + \sigma_Q\rho_2 = \left[\frac{\epsilon\left(1-\rho^2\right)}{(\kappa+\delta)}\left(\alpha - \frac{1}{2}l^2 - \frac{1}{2\delta}\right) + l\rho_2\right]\sqrt{Y_{2t}}\,,$$

implying that the sign of $V_{Y_2}\sigma_{Y_2}(1-\rho^2) + \sigma_Q\rho_2$ is given by the sign of $\left[\frac{\epsilon(1-\rho^2)}{(\kappa+\delta)}\left(\alpha - \frac{1}{2}l^2 - \frac{1}{2\delta}\right) + l\rho_2\right]$. The threshold value ω in (26) and (28) is therefore obtained:

$$\begin{aligned} 0 &= \frac{\epsilon\left(1-\rho^2\right)}{(\kappa+\delta)}\left(\alpha - \frac{1}{2}l^2 - \frac{1}{2\delta}\right) + l\rho_2 \\ \Leftrightarrow \alpha &= \underbrace{\frac{\left(l^2\delta\epsilon + \epsilon\right)\left(1-\rho^2\right) - 2\delta(\kappa+\delta)\,l\rho_2}{2\delta\epsilon\left(1-\rho^2\right)}}_{\omega}. \end{aligned}$$

When $\alpha > \omega\,(\alpha < \omega)$ then $V_{Y_2}\sigma_{Y_2}(1-\rho^2) + \sigma_Q\rho_2 > 0\,(< 0)$, making it immediate to obtain (26) from (15) and (28) from (16).

□

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