Industry Dynamics and Aggregate Stability over Transition

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This paper presents an endogenous growth model of directed technical change with vertical and horizontal R&D to study an analytical mechanism that is consistent with the coexistence of structural change and aggregate stability. We focus on shifts in the share of the high- vis-à-vis the low-tech sectors within manufacturing in the context of slow, but flexible, transitional dynamics, which arises in the context of a three-dimensional stable manifold. Under the hypothesis of a positive shock in the proportion of high-skilled labour, the technological-knowledge bias channel leads to nonbalanced sectoral growth, while the aggregate variables remain approximately constant and thus consistent with the Kaldor facts. With prevailing market-scale effects, a calibration exercise shows that the model is able to account for around two-thirds of the increase in the share of the high-tech sectors observed in European data from 1995 to 2007.

Keywords: industry dynamics, Kaldor facts, high tech, low tech, directed technical change

JEL Classification: O41, O31

1. Introduction

During the last century, there is evidence that developed countries have been satisfying two types of stylised facts: Kuznets and Kaldor facts. The former state that structural change takes place over time due to nonbalanced sectoral growth (Kongsamut, Rebelo, and Xie, 2001; Gollin, Parente, and Rogerson, 2002; Greenwood and Uysal, 2005). The latter point out that the economic growth rate, the capital-output ratio, the real interest

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rate and the labor income share are fairly stable over long periods of time (Kaldor, 1961; Kongsamut, Rebele, and Xie, 2001), thus conveying the idea of balanced aggregate growth and of aggregate stability. While the Kaldor facts have become a benchmark in growth theory, implying that a successful model should comply with them, the Kuznets facts have drawn less attention within that literature.

From a theoretical perspective, the coexistence of Kuznets and Kaldor facts seems puzzling. To generate features of nonbalanced sectoral growth that are consistent with balanced aggregate growth and aggregate stability, a promising literature on structural change has recently been developed. In particular, a strand of the literature (e.g., Ngai and Pissarides, 2007; Bonatti and Felice, 2008; Blankenau and Cassou, 2009) has focused on the supply side, analysing the cross-sector differences in Total Factor Productivity (TFP) growth rates and their implication for the dynamics of sectoral input reallocation. From this framework, the authors derive an equilibrium growth path that features simultaneously balanced growth of aggregate variables and nonbalanced growth of sectoral variables. Two particular features of this literature stand out. First, the aggregate stability observed in the data is addressed from the perspective of the balanced-growth-path (BGP) equilibrium, usually derived under knife-edge conditions, while, at the same time, the nonbalanced sectoral growth may induce an ever increasing (decreasing) share of the sector with higher (lower) TFP growth. Second, these models attempt to account for structural change viewed as the reallocation of resources between the “technologically progressive” sector (i.e., the high TFP growth sector), usually identified empirically with manufacturing, and the “ stagnant” sector (i.e., the low TFP growth sector), usually agriculture or services.

However, the shifts of economic activity within the “technologically progressive” sector seem to have deserved less attention. Our paper focuses on structural change, or industry dynamics, associated with shifts in the share of the high- vis-à-vis the low-tech sectors within manufacturing over time and on its compatibility with stable aggregate variables. A similar approach to structural change is followed by Acemoglu and Guerrieri (2008), who however focus on high versus low capital intensity sectors. Moreover, we address the interplay between aggregate stability and structural change in the context of slow transitional dynamics: as the economy evolves towards the BGP, there is a shift of resources from the high- (low-)tech sectors to the low- (high-)tech ones, while the aggregate variables remain approximately constant – i.e., being almost indistinguishable from a true BGP as far as these variables are concerned. When the BGP is (asymptotically) reached, balanced growth at both the aggregate and the sectoral level is established, and thus no sector ever vanishes. Aggregate stability tackled from the perspective of slow transitional dynamics can be found in Kongsamut, Rebele, and Xie (2001, Appendix) and Acemoglu and Guerrieri (2008, Section III). In particular, the latter emphasise the importance of looking into transitional dynamics, instead of the BGP, in as much as it may produce (approximate) aggregate stability simultaneously with structural change in a situation.

An other relevant strand of literature on structural change has focused on the demand-side factors explaining nonbalanced growth (e.g., Kongsamut, Rebele, and Xie, 2001; Foellmi and Zweimüller, 2008).
in which all sectors have nontrivial shares, as seems to be the case empirically.

Under our framework, two specific measures of industry structure are of interest: the *relative* number of firms and *relative* production, i.e., the number of firms and production in the high- versus the low-tech manufacturing sectors. However, it is not a priori clear how these measures should relate to each other. Figure 1 depicts the time-series data for relative production (over the 1980-2007 period) and the relative number of firms (over 1995-2007) for 14 European countries. In order to compare more finely the behaviour of relative production and the relative number of firms over time, we considered the longest period with available data for both variables (1995-2007) and computed their cross-country weighted average. We found that the average annual growth rate was positive for both relative production and the relative number of firms, but that the former exceeded the latter by 0.52 percentage points/year (1.22 percent versus 0.7 percent). In the period 1995-2000, both variables grew at a faster pace (2.8 percent versus 1.06 percent) and the drift between them was also larger (1.74 percentage points/year).

Thus, although the time-span of the available data is relatively short, the presented anecdotal evidence suggests that industry dynamics has been broadly characterised by an increase in the share of the high-tech sectors, and that has mainly occurred in terms of production, hence implying an increase in average firm size in the high- vis-à-vis the low-tech sectors. Taking a longer term perspective, we conjecture that this disparity is due to the asymmetric role played by the extensive and the intensive margin of industrial growth across the high- and the low-tech sectors, where the former pertains to the creation of new products/firms and the latter to the increase of product quality of existing products and, thereby, of production per firm. Therefore, although in line with the general view that industrial growth proceeds both along an intensive and an extensive margin in the very long run (e.g., Freeman and Soete, 1997), we expect a rich interaction between the two margins for shorter time horizons, namely in response to structural shocks. Having in mind (i) the observed specificity of the high- and low-tech sectors regarding the proportion of high-skilled labour, (ii) the swift change in the proportion of high skilled workers in manufacturing found in the data between the 80’s and the 90’s across a number of developed countries (see, e.g., Acemoglu, 2003) and (iii) the acceleration of relative production through the 90’s (see the upper panel in Figure 1), we emphasise in

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2The source is the Eurostat on-line database on Science, Technology and Innovation, available at [http://epp.eurostat.ec.europa.eu](http://epp.eurostat.ec.europa.eu), where the OECD classification of high- and low-tech sectors is used (see Hatzichronoglou, 1997). High-tech sectors are aerospace, computers and office machinery, electronics and communications, and pharmaceuticals, while the low-tech sectors are petroleum refining, ferrous metals, paper and printing, textiles and clothing, wood and furniture, and food and beverages. By crossing the data on both variables – production and the number of firms – and considering a minimum time-span of 12 years (which is the maximum time span available for the number of firms), we end up with a sample of 14 European countries, as depicted by Figure 1. For the best of our knowledge, the Eurostat on-line database is the only one with available data for the number of firms in manufacturing broke down according to the referred to OECD classification.

3Empirical evidence suggests that high-tech sectors are more intensive in high-skilled labour than the low-tech sectors. For instance, according to the data for the average of the European Union (27 countries), 30.9% of the employment in the high-tech manufacturing sectors is high skilled ("college graduates"), against 12.1% of the employment in the low-tech sectors. The source is the Eurostat on-line database on Science, Technology and Innovation ([http://epp.eurostat.ec.europa.eu](http://epp.eurostat.ec.europa.eu)).
particular the hypothesis of a shock in the form of an increase in the relative supply of skills (i.e., the ratio of high- to low-skilled workers) and its asymmetric impact on the intensive and the extensive margin, both within and across the high- and the low-tech sectors, through the technological-knowledge bias channel. As explained further below, the different nature of the intensive and the extensive margin should play a central role here.

To uncover the analytical mechanism through which the empirical evidence can be accommodated, we propose a general equilibrium growth model that incorporates endogenous directed technical change with vertical R&D (increase of product quality) and horizontal R&D (creation of new products/industries). Final-goods production uses either low- or high-skilled labour with labour-specific intermediate goods, while R&D can be directed to either the low- or the high-skilled labour complementary technology. Thus, “sector” herein represents a group of firms producing the same type of labour-complementary intermediate goods. Since the data shows that the high-tech sectors are more intensive in high-skilled labour than the low-tech sectors (see fn. 3), we consider the high- and low-skilled labour-complementary intermediate-good sectors in the model as the theoretical counterpart of the high- and low-tech sectors (e.g., Cozzi and Impullitti, 2010). The adopted directed technical change setup implies constant TFP growth rates across sectors along the BGP; it is in this sense that, in the light of our model, the high- and the low-tech sectors are both regarded as “technologically progressive” sectors.

We consider an R&D specification that implies that the choice between vertical and horizontal innovation is related to the splitting of R&D expenditures, which are fully endogenous. Therefore, we endogenise the rate of both intensive and extensive growth, and thereby the number of industries/firms in each sector. Given the distinct nature of vertical and horizontal innovation (immaterial versus physical) and the consequent asymmetry in terms of R&D complexity and congestion costs, vertical R&D emerges as the ultimate growth engine, while horizontal R&D builds an explicit link between aggregate and industry-structure variables (the number of firms and production in high- and in low-tech sectors).

Furthermore, we take a flexible view of scale effects on industrial growth: the complete removal of scale effects as sometimes posited in the theoretical growth literature is a knife-edge case, as Peretto and Smulders (2002) have recently stressed. Indeed, the existence of scale effects at the aggregate level is disputed, with the empirical results only unequivocally rejecting it in secular trend (e.g., Jones, 1995b), whereas empirical studies clearly indicate the existence of scale effects at the industry (manufacturing)

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4 The relative supply of skills is usually treated as exogenous in the literature of directed technical change, in order to isolate the impact of the increase of the proportion of the high-skilled workers observed in the data through the technological knowledge bias mechanism (e.g., Acemoglu, 2002, 2003).

5 An alternative approach in the literature assumes that the allocation of resources between vertical and horizontal R&D implies a division of labour between the two types of R&D. Since the total labour level is determined exogenously, the rate of growth along the horizontal direction is exogenous, i.e., the BGP flow of new products and industries occurs at the same rate as (or is proportional to) population growth.
Figure 1: The share of the high-tech sectors through time: relative production (upper panel) and the relative number of firms (lower panel) according to the high-tech low-tech OECD classification. Source: Eurostat on-line database on Science, Technology and Innovation – table “Economic statistics on high-tech industries and knowledge-intensive services at the national level”, available at http://epp.eurostat.ec.europa.eu.
level (e.g., Backus, Kehoe, and Kehoe, 1992). Thus, because the literature does not offer a clear cut answer to the issue of the existence of scale effects, we consider a number of scenarios, from no scale effects on growth (only price-channel effects exist) to full scale effects (only market-size-channel effects exist). This will then allow for a flexible relationship between the number of firms, production and firm size across the high- and the low-tech sectors.

As for the methodology, we focus on global transitional dynamics. Global dynamics, as opposed to local dynamics, allows us to carry out a comparative dynamics exercise without restricting the analysis to an arbitrarily close neighbourhood of the steady state and, thus, to arbitrarily small shifts in the parameters and the exogenous variables. Since the dynamic system in our model is four dimensional (in appropriately detrended variables), with three pre-determined endogenous variables, and is highly non linear, we resort to numerical methods to study global dynamics. In particular, the model is solved by numerical integration using a finite difference method implementing the three-stage Lobatto IIIa formula. The code performs a mesh selection and error control based on the residual of the continuous solution and is provided through the software MatLab.

We analyse transitional dynamics by considering the effects of an unanticipated one-off shock in the relative supply of skills. An interesting asymmetry between the high- and the low-tech sectors then arises working through the technological-knowledge bias channel, because of the difference in profitability between those two sectors induced by the initial rise in the proportion of high-skilled labour: under prevailing market-size-channel effects (price-channel effects), the vertical innovation rate targeting the low-tech sector experiences an immediate decrease (increase) while the rate in the high-tech sector takes an upward (downward) jump; then, given the complementarity between vertical and horizontal R&D, this sets off an asymmetric adjustment of both the vertical and the horizontal innovation rate – and hence of growth rates along the intensive and extensive margin – across sectors over the medium-run. As the economy slowly adjusts towards the new BGP, industry dynamics coexists with aggregate stability. There is a significant shift of economic activity between sectors due to the nonbalanced industrial growth; however, at the aggregate level, the economic growth rate and the real interest rate remain approximately constant, which implies that the model approximates the Kaldor facts along transitional dynamics.

Moreover, our model implies a speed of convergence to the new BGP that is faster at the sectoral than at the aggregate level, in particular if one compares the share of the high-tech sectors in production with the economic growth rate. More generally, transitional dynamics is flexible in the sense that the transition speed is different both across variables and through time, even if the time-paths are derived from a linearised dynamic system, which reflects the existence of a multi-dimensional stable manifold. Such a result was firstly explored within an endogenous-growth setup by Eicher and Turnovsky (2001). However, while in the latter a multi-dimensional stable manifold arises from the removal of a tolia, Chatterjee, and J. Turnovsky (2010) investigate the reliability of employing linearisation to evaluate the transitional dynamics in neoclassical growth models and conclude that, when transition is slow – as is the case in our model –, linearisation tends to yield misleading predictions.
of scale effects in a Jones (1995a)-type model, we derive our results under less strict conditions with this respect, since scale-effect removal is not a necessary condition.\footnote{Eicher and Turnovsky (2001) analyse the dynamics of an endogenous growth model with physical capital and horizontal R&D, in which labour is the input, based on Jones (1995a), and show that the removal of scale effects in that type of models raises the dimension of the dynamic system such that the latter becomes four-dimensional and the stable manifold two-dimensional. In our model of vertical and horizontal R&D and two intermediate-good sectors, where the homogeneous final-good is the input to R&D activities, we are able to derive a four-dimensional dynamic system featuring a three-dimensional stable manifold irrespective of the degree of scale effects.}

In the case of prevailing market-scale channel effects, the theoretical results are consistent with the time-series data depicted by Figure 1. That is, there is an increase in the share of the high-tech sectors both in terms of production and of the number of firms, paralleled by a discrete increase in production per firm relatively to the low-tech sectors. The former result stems from the positive response of the two measures of industry structure to the shock through the technological-knowledge bias channel (a larger market, measured by employed high-skilled labour, expands profits and, thus, the incentives to allocate resources to both types of R&D in the high-tech sectors), while the latter is explained by the stronger complexity and congestion costs impinging on horizontal R&D, which slow down and dampen the response of the number of firms relatively to that of production. According to a simple calibration exercise, the model is able to account for around two-thirds of the increase in the share of the high-tech sectors observed in the European data from 1995 to 2007.

Regarding the compatibility of our calibration results with the referred to debate over the counterfactual character of scale effects on growth, we note that the existing empirical results reject the existence of scale effects in secular trend, as cited earlier, while our quantitative results suggest the existence of scale effects pertaining to the medium term behaviour of the economies – in particular in the light of the relatively short time span of the time-series data that support our calibration exercise. In this sense, our results seem to be complementary to the long-term vision of industrial growth as a non-scale phenomenon.

The remainder of the paper has the following structure. In Section 2, we present the model of directed technological change with vertical and horizontal R&D, derive the dynamic general equilibrium and characterise the BGP. In Section 3, we detail the comparative dynamics results by considering the impact of a shock in the relative supply of skills on the aggregate and the industry-level variables, and carry out an illustrative calibration exercise. Section 4 gives some concluding remarks.

2. The model

The model used herein is drawn from Acemoglu and Zilibotti (2001), augmented with vertical R&D, as introduced in Afonso (2006), and developed under flexible scale effects. Thus, we study a directed technological change model with vertical and horizontal R&D, built into a dynamic general equilibrium setup of a closed economy where the aggregate competitively-produced final good can be used in consumption, production of
intermediate goods and R&D. The economy is populated by a fixed number of infinitely-lived households who inelastically supply one of two types of labour to final-good firms: low-skilled, \( L \), and high-skilled labour, \( H \). The final good is produced by a continuum of firms, indexed by \( n \in [0,1] \), to whom two substitute technologies are available: the “Low” (respectively, “High”) technology uses a combination of \( L \) (\( H \)) and a continuum of \( L \)-specific (\( H \)-specific) intermediate goods indexed by \( \omega_L \in [0,N_L] \) (\( \omega_H \in [0,N_H] \)).

Potential entrants can devote resources to either horizontal or vertical R&D, and directed to either the high- or the low-skilled labour-complementary technology. Horizontal R&D increases the number of industries, \( N_m, m \in \{L,H\} \), in the \( m \)-complementary intermediate-good sector,\(^8\) while vertical R&D increases the quality level of the good of an existing industry, indexed by \( j_m(\omega_m) \). Then, the quality level \( j_m(\omega_m) \) translates into productivity of the final producer by using the good produced by industry \( \omega_m, \lambda^{j_m(\omega_m)} \), where \( \lambda > 1 \) is a parameter measuring the size of each quality upgrade. By improving on the current best quality index \( j_m \), a successful R&D firm will introduce the leading-edge quality \( j_m(\omega_m) + 1 \) and hence render inefficient the existing input. Therefore, the successful innovator will become a monopolist in \( \omega_m \). However, this monopoly, and the monopolist earnings that come with it, are temporary, because a new successful innovator will eventually substitute the incumbent.

**2.1. Production and price decisions**

This section briefly describes the familiar components of Acsenoglu and Zilibotti’s (2001) model, augmented with vertical R&D. Aggregate output at time \( t \) is defined as \( Y_{tot}(t) = \int_0^1 P(n,t)Y(n,t)dn \), where \( P(n,t) \) and \( Y(n,t) \) are the relative price and the quantity of the final good produced by firm \( n \). Each final-good firm \( n \) has a constant-returns-to-scale technology using low- and high-skilled labour and a continuum of labour-specific intermediate goods with measure \( N_m(t) \), such that \( N_{tot}(t) = N_L(t) + N_H(t) \) and

\[
Y(n,t) = A \left[ \int_0^{N_L(t)} (\lambda^{j_L(\omega_L,t)} \cdot X_L(n,\omega_L,t))^{1-\alpha} d\omega_L \right] [(1-n) \cdot l \cdot L(n)]^\alpha + A \left[ \int_0^{N_H(t)} (\lambda^{j_H(\omega_H,t)} \cdot X_H(n,\omega_H,t))^{1-\alpha} d\omega_H \right] [n \cdot h \cdot H(n)]^\alpha, \quad 0 < \alpha < 1,
\]

(1)

where \( A > 0 \) is the total factor productivity, \( L(n) \) and \( H(n) \) are the labour inputs used by \( n \) and \( \alpha \) is the labour share in production, and \( \lambda^{j_m(\omega_m,t)} \cdot X_m(n,\omega_m,t) \) is the input of \( m \)-complementary intermediate good \( \omega_m \) measured in efficiency units at time \( t \).\(^9\) An absolute-productivity advantage of \( H \) over \( L \) is captured by \( h > l \geq 1 \); a relative-productivity advantage of each labour type is determined by terms \( n \) and \( (1-n) \), implying that \( H \) is relatively more productive for larger \( n \), and vice-versa. As explained below, at each \( t \) there is a competitive equilibrium threshold \( \bar{n}(t) \), endogenously determined, where the switch from one technology to the other becomes advantageous, so that each \( n \) produces exclusively with one technology, either \( L \)- or \( H \)-technology.

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\(^8\)Henceforth, we will refer to the “\( m \)-complementary intermediate-good sector” as “\( m \)-technology sector”.

\(^9\)In equilibrium, only the top quality of each \( \omega_m \) is produced and used; thus, \( X_m(j,\omega_m,t) = X_m(\omega_m,t) \).
Final producers take the price of their final good, \( P(n, t) \), wages, \( w_m(t) \), and input prices \( p_m(\omega_m, t) \) as given. From the profit maximisation conditions, we determine the demand of intermediate good \( \omega_m \) by firm \( n \)

\[
X_L(n, \omega_L, t) = (1 - n) \cdot l \cdot L(n) \cdot \left( \frac{A \cdot P(n,t) \cdot (1 - \alpha)}{p_L(\omega_L, t)} \right)^{\frac{1}{\alpha}} \chi^L(\omega_L, t) \chi^{\frac{1}{\alpha}}(\omega_L, t) \quad (2)
\]

\[
X_H(n, \omega_H, t) = n \cdot h \cdot H(n) \cdot \left( \frac{A \cdot P(n,t) \cdot (1 - \alpha)}{p_H(\omega_H, t)} \right)^{\frac{1}{\alpha}} \chi^H(\omega_H, t) \chi^{\frac{1}{\alpha}}(\omega_H, t)
\]

There is monopolistic competition if we consider the whole sector: the monopolist in industry \( \omega_m \in [0, N_m(t)] \) fixes the price \( p_m(\omega_m, t) \) but faces an isoelastic demand curve,

\[
X_L(\omega_L, t) = \int_0^{\bar{n}(t)} X_L(n, \omega_L, t) dn \quad \text{or} \quad X_H(\omega_H, t) = \int_{\bar{n}(t)}^1 X_H(n, \omega_H, t) dn \quad (\text{see } (2))
\]

We assume that intermediate goods are non-durable and entail a unit marginal cost of production, measured in terms of the final good, whose price is taken as given (nominal).

Profit in \( \omega_m \) is thus \( \pi_m(\omega_m, t) = (p_m(\omega_m, t) - 1) \cdot X_m(\omega_m, t) \), and the profit maximising price is a constant markup over marginal cost

\[
p_m(\omega_m, t) \equiv p = \frac{1}{1 - \alpha} > 1, \ m \in \{L, H\}
\]

Given \( \bar{n} \) and (3), we can then write the final-good output as

\[
Y(n, t) = \begin{cases}
A \bar{n} P(n, t) \left( \frac{1 - \alpha}{\alpha} \right)^{\frac{2(1 - \alpha)}{\alpha}} \cdot (1 - n) \cdot l \cdot L(n) \cdot Q_L(t) & , 0 \leq n \leq \bar{n} \\
A \bar{n} P(n, t) \left( \frac{1 - \alpha}{\alpha} \right)^{\frac{2(1 - \alpha)}{\alpha}} \cdot n \cdot h \cdot H(n) \cdot Q_H(t) & , \bar{n} \leq n \leq 1
\end{cases}
\]

where the aggregate quality index

\[
Q_m(t) = \int_0^{N_m(t)} q_m(\omega_m, t) d\omega, \quad q_m(\omega_m, t) \equiv \chi^{\frac{1}{m_m(\omega_m, t)}}(\frac{1 - \alpha}{\alpha}), \ m \in \{L, H\}
\]

measures the technological-knowledge level in each \( m \)-technology sector. Thus, \( Q \equiv Q_H/Q_L \) measures the technological-knowledge bias. The allocation of the low- and high-skilled labour inputs to the \( L \)- and the \( H \)-technology sector verifies \( L = \int_0^\bar{n} L(n) dn \) and \( H = \int_0^1 H(n) dn \). With competitive final-good producers, economic viability of either \( L \)- or \( H \)-technology relies on the relative productivity and price of labour, as well as on the relative productivity and prices of intermediate goods, due to complementarity in production. Labour prices depend on quantities, \( H \) and \( L \). In relative terms, the productivity-adjusted quantity of \( H \) is \( \mathcal{H}/\mathcal{L} \), where \( \mathcal{H} \equiv hH \) and \( \mathcal{L} \equiv lL \). As for the productivity and prices of intermediate goods, they depend on complementarity with either \( H \) or \( L \), on the technological knowledge embodied and on the markup. These determinants are summed up in \( Q_L \) and \( Q_H \). The endogenous threshold \( \bar{n} \) follows from equilibrium in the inputs markets, and relies on the determinants of economic viability of the two technologies, such that

\[
\bar{n}(t) = \left[ 1 + \left( \frac{\mathcal{H} Q_H(t)}{\mathcal{L} Q_L(t)} \right)^{\frac{1}{2}} \right]^{-1}
\]
\( \bar{n}(t) \) implies that \( L\)-(\( H\))-complementary technology is exclusively used by final-good firms indexed by \( n \in [0, \bar{n}(t)] \) \((n \in [\bar{n}(t), 1])\), and it can be related to the ratio of price indices of final goods produced with \( L\)- and \( H\)-technologies:

\[
\frac{P_H(t)}{P_L(t)} = \left( \frac{\bar{n}(t)}{1 - \bar{n}(t)} \right)^\alpha,
\]

where

\[
\begin{align*}
P_L(t) &= P(n, t) \cdot (1 - n)^\alpha = e^{\alpha \bar{n}(t)^{-\alpha}} \quad n \in [0, \bar{n}(t)], \\
P_H(t) &= P(n, t) \cdot n^\alpha = e^{\alpha(1 - \bar{n}(t))^{-\alpha}}.
\end{align*}
\]  

(7)

In (7), we first define the price indices, \( P_L(t) \) and \( P_H(t) \), by recognising that, in equilibrium, the marginal value product, \( \frac{d m(n)}{d m(n)} P(n, t) Y(n, t) \), must be constant over \( n \), implying that \( P(n, t)^{\frac{\alpha}{2}} \cdot (1 - n) \) and \( P(n, t)^{\frac{\alpha}{2}} \cdot n \) must be constant over \( n \in [0, \bar{n}(t)] \) and \( n \in [\bar{n}(t), 1] \), respectively. Then, considering that at \( \bar{n}(t) \) the \( L\)- and \( H\)-technology firms must break even, we relate \( P_L(t) \) and \( P_H(t) \) with \( \bar{n}(t) \). Equation (6) shows that if either the technology is highly \( H\)-biased or if there is a large relative supply of \( H\), the share of final goods using the \( H\)-technology is large and \( \bar{n}(t) \) is small. By (7), small \( \bar{n}(t) \) implies a low \( P_H(t)/P_L(t) \). In this case, the demand for \( \omega_H \in [0, N_H(t)] \) is low, which discourages R&D activities directed to \( H\)-technology.

From (2), (3) and (7), we find the optimal intermediate-good production, \( X_m(\omega_m) \), and thus the optimal profit accrued by the monopolist in \( \omega_m \) is

\[
\pi_m(\omega_m, t) = \pi_{0m} \cdot P_m(t)^{\frac{1}{\alpha}} \cdot q_m(\omega_m, t), \quad m \in \{L, H\},
\]

(8)

where \( \pi_{0L} \equiv L A^{\frac{1}{\alpha}} \left( \frac{\alpha}{1 - \alpha} \right)^{\frac{2}{\alpha}} \) and \( \pi_{0H} \equiv H A^{\frac{1}{\alpha}} \left( \frac{\alpha}{1 - \alpha} \right)^{\frac{2}{\alpha}} \) are positive constants.

Total intermediate-good optimal production, \( X_{tot}(t) \equiv X_L(t) + X_H(t) \equiv \int_0^{N_L(t)} X_L(\omega_L)d\omega_L + \int_0^{N_H(t)} X_H(\omega_H)d\omega_H \), and total final-good optimal production, \( Y_{tot}(t) \equiv Y_L(t) + Y_H(t) \equiv \int_0^{\bar{n}(t)} P(n, t) Y(n, t)dn + \int_{\bar{n}(t)}^1 P(n, t) Y(n, t)dn \), are, respectively,

\[
X_{tot}(t) = \chi_X \Gamma(t)
\]

(9)

and

\[
Y_{tot}(t) = \chi_Y \Gamma(t).
\]

(10)

where \( \chi_X \equiv A^{\frac{1}{\alpha}} (1 - \alpha)^{\frac{2}{\alpha}} \), \( \chi_Y \equiv A^{\frac{1}{\alpha}} (1 - \alpha)^{\frac{2(1 - \alpha)}{\alpha}} \) and \( \Gamma(t) \equiv P_L(t)^{\frac{1}{\alpha}} \cdot \mathcal{L} \cdot Q_L(t) + P_H(t)^{\frac{1}{\alpha}} \cdot \mathcal{H} \cdot Q_H(t) \).

2.2. R&D

We consider two R&D sectors, one targeting horizontal innovation and the other endeavoring vertical innovation. We assume that the pools of innovators performing the two types of R&D are different. Each new design (a new variety or a higher quality good) is granted a patent and thus a successful innovator retains exclusive rights over the use of his/her good. We also take the simplifying assumptions that both vertical and horizontal R&D are performed by (potential) entrants, and that successful R&D leads to the set-up
of a new firm in either an existing or in a new industry (e.g., Howitt, 1999; Strulik, 2007; Gil, Brito, and Afonso, 2013). There is perfect competition among entrants and free entry in R&D business.

2.2.1. Vertical R&D

By improving on the current top quality level \( j_m(\omega_m, t) \), a successful R&D firm earns monopoly profits from selling the leading-edge input of \( j_m(\omega_m, t) + 1 \) quality to final-good firms. A successful innovation will instantaneously increase the quality index in \( \omega_m \) from \( q_m(\omega_m, t) = q_m(j_m) \) to \( q_m(\omega_m, t) = q_m(j_m + 1) = \lambda^{(1-\alpha)/\alpha} q_m(j_m) \). In equilibrium, lower qualities of \( \omega_m \) are priced out of business.

Let \( I_m^i(j_m) \) denote the Poisson arrival rate (vertical-innovation rate) by potential entrant \( i \) in industry \( \omega_m \) when the highest quality is \( j_m \). The rate \( I_m^i(j_m) \) is independently distributed across firms, across industries and over time, and depends on the flow of resources \( R_{im}(j_m) \) committed by entrants at time \( t \). As in, e.g., Barro and Sala-i-Martin (2004, ch. 7), \( I_m^i(j_m) \) features constant returns in R&D expenditures, \( I_m^i(j_m) = R_{im}(j_m) \cdot \Phi_m(j_m) \), where \( \Phi_m(j_m) \) is the R&D productivity factor, which is assumed to be homogeneous across \( i \) in \( \omega_m \). We assume

\[
\Phi_L(j_L) = \frac{1}{\zeta \cdot q_L(j_L + 1) \cdot L^\epsilon} \quad \text{and} \quad \Phi_H(j_H) = \frac{1}{\zeta \cdot q_H(j_H + 1) \cdot H^\epsilon},
\]

where \( \zeta \equiv \zeta_L \equiv \zeta_H > 0 \) is a constant (flow) fixed vertical-R&D cost, and \( \epsilon \geq 0 \). Hence, a complexity effect is considered (e.g., Barro and Sala-i-Martin, 2004, ch. 7; Etro, 2008), implying vertical-R&D dynamic decreasing returns to scale (decreasing returns to cumulated R&D): the larger the level of quality, \( q_m \), the costlier it is to introduce a further jump in quality.\(^{10}\) Equation \((11)\) also implies that an increase in market scale, \( L \) or \( H \), dilutes the effect of R&D outlays on innovation probability; this captures the idea that the difficulty of introducing new qualities and replacing old ones is proportional to the market size measured by employed labour in efficiency units (e.g., Barro and Sala-i-Martin, 2004), due to coordination, organisational and transportation costs and rental protection actions by incumbents (e.g., Dinopoulos and Thompson, 1999; Sener, 2008).

Depending on the effectiveness of those costs and actions, they may partially (\( 0 < \epsilon < 1 \)), totally (\( \epsilon = 1 \)) or over (\( \epsilon > 1 \)) counterbalance the scale benefits on profits, which accrue to the R&D successful firm at each \( t \). Thus, there may be, respectively, positive, null or negative net scale effects on growth, as measured by \( 1 - \epsilon \).\(^{11}\) Aggregating across \( i \) in \( \omega_m \),

\(^{10}\) The way \( \Phi \) depends on \( j \) implies that the increasing difficulty of creating new product generations over \( t \) exactly offsets the increased rewards from marketing higher quality products; see \((11)\) and \((8)\). This allows for constant vertical-innovation rate over \( t \) and across \( \omega \) in BGP (on asymmetric equilibrium in quality-ladders models and its growth consequences, see Cozzi, 2007).

\(^{11}\) Sener (2008) contrasts the effects of rental protection actions with the expanding variety and the dynamic decreasing returns to R&D as scale-removal mechanisms within a quality-ladders model with knowledge-driven R&D specification. Observe, however, that the dynamic decreasing returns to R&D, as first introduced by Segerstrom (1998), and represented in our model by the term \( 1/q_m(j_m+1) \) in \((11)\), are neither necessary nor sufficient for the purpose of scale removal in a model with lab-equipment specification (though it plays a crucial role in guaranteeing a Poisson rate constant over
we get $R_{vm}(j_m) = \sum_i R^i_{vm}(j_m)$ and $I_m(j_m) = \sum_i I^i_m(j_m)$, and thus

\[ I_L(j_L) = R_{vL}(j_L) \cdot \Phi_L(j_L) \quad \text{and} \quad I_H(j_H) = R_{vH}(j_H) \cdot \Phi_H(j_H). \tag{12} \]

As the terminal date of each monopoly arrives as a Poisson process with frequency $I_m(j_m)$ per (infinitesimal) increment of time, the present value of a monopolist’s profits is a random variable. Let $V_m(j_m)$ denote the expected value of an incumbent firm with current quality level $j_m(\omega_m, t)$, $^{12}$

\[ V_m(j_m) = \pi_0 m q_m(j_m) \int_{t}^{\infty} P_m(t) \frac{1}{\zeta} \cdot e^{-r_v(r_v + I_m(j_m))dt} ds, \quad m \in \{L, H\}, \tag{13} \]

where $r_v$ is the equilibrium market real interest rate, and $\pi_0 m q_m(j_m) = \pi_m(j_m) P_m^{-\frac{1}{\alpha}}$, given by (8) and (7), is constant in-between innovations. Free-entry prevails in vertical R&D such that the condition $I_m(j_m) \cdot V_m(j_m + 1) = R_{vm}(j_m)$ holds, which implies that

\[ V_L(j_L + 1) = \frac{1}{\Phi_L(j_L)} \quad \text{and} \quad V_H(j_H + 1) = \frac{1}{\Phi_H(j_H)} . \tag{14} \]

Next, we determine $V_m(j_m + 1)$ analogously to (13), then consider (14) and time-differentiate the resulting expression. Thus, if we also consider (8), we get the no-arbitrage condition facing a vertical innovator

\[ r(t) + I_L(t) = \frac{\pi_0 \cdot L^{1-\epsilon} \cdot P_L(t)^{\frac{1}{\alpha}}}{\zeta}, \quad r(t) + I_H(t) = \frac{\pi_0 \cdot H^{1-\epsilon} \cdot \cdot P_H(t)^{\frac{1}{\alpha}}}{\zeta}, \tag{15} \]

where $\pi_0 \equiv \pi_0 L / L = \pi_0 H / H$. It has two implications: the present value of “basic” profits $\pi_0 \cdot L^{1-\epsilon} \cdot P_L(t)^{\frac{1}{\alpha}}$ and $\pi_0 \cdot H^{1-\epsilon} \cdot \cdot P_H(t)^{\frac{1}{\alpha}}$, $^{14}$ using the effective rate of interest $r(t) + I_m(t)$ as a discount factor, should be equal to the fixed cost of entry; and the rates of entry are symmetric across industries $I_m(\omega_m, t) = I_m(t)$.

If we equate the effective rate of return for both R&D sectors by considering (15), the no-arbitrage condition obtains

\[ I_H(t) - I_L(t) = \frac{\pi_0 \cdot (H^{1-\epsilon} \cdot \cdot P_H(t)^{\frac{1}{\alpha}} - L^{1-\epsilon} \cdot P_L(t)^{\frac{1}{\alpha}})}{\zeta} \quad . \tag{16} \]

$\omega_m$ and hence the existence of a symmetric equilibrium; see fn. 10). The same applies to the expanding variety mechanism, as it is clear if we let $\epsilon = 0$ in our results below.

$^{12}$We assume that entrants are risk-neutral and, thus, only care about the expected value of the firm.

$^{13}$Observe that, from (8) and (12), we have $\frac{\pi_0 m (\omega_m, t)}{\pi_0 m (\omega_m, \tau)} - \frac{1}{\alpha} P_m(t) = I_m(\omega_m, t) \cdot \left[ j_m(\omega_m, t) \cdot \left( \frac{\alpha}{1-\alpha} \right) \right] \cdot \ln \lambda$. Thus, if we time-differentiate (14) considering (13) and the equations above, we get $r(t) = \frac{\pi_0 m (j_m, t)}{\pi_0 m (j_m, j_m, t)} - I_m(j_m + 1)$, which can then be re-written as (15).

$^{14}$These are the profit flows that accrue when $j_m = 0$ (i.e., $q_m = 1$).
Solving (12) for \( R_{vm}(\omega_m, t) = R_{vm}(j_m) \) and aggregating across industries \( \omega_m \), we determine total resources devoted to vertical R&D, \( R_{vm}(t) \); e.g., with \( m = L \), \( R_{vL}(t) = \int_0^{N_L(t)} R_{vL}(\omega_L, t) \, d\omega_L = \int_0^{N_L(t)} \zeta \cdot L^c \cdot q^{Lc}_L(\omega_L, t) \cdot I_L(\omega_L, t) \, d\omega_L \). As the innovation rate is industry independent, then

\[
R_{vL}(t) = \zeta \cdot L^c \cdot \lambda^{\frac{1+\alpha}{\alpha}} \cdot I_L(t) \cdot Q_L(t), \quad R_{vH}(t) = \zeta \cdot H^c \cdot \lambda^{\frac{1+\alpha}{\alpha}} \cdot I_H(t) \cdot Q_H(t). \tag{17}
\]

### 2.2.2. Horizontal R&D

Variety expansion arises from R&D aimed at creating a new intermediate good. Again, innovation is performed by a potential entrant, which means that, because there is free entry, the new good is produced by new firms. Under perfect competition among R&D firms and constant returns to scale at the firm level, instantaneous entry is obtained as \( \dot{N}_m(t) = R_{hm}(t)/\eta_m(t) \), where \( \dot{N}_m(t) \) is the contribution to the instantaneous flow of new \( m \)-complementary intermediate goods by R&D firm \( e \) at a cost of \( \eta_m(t) \) units of the final good (cost of horizontal entry) and \( R_{hm}(t) \) is the flow of resources devoted to horizontal R&D by innovator \( e \) at time \( t \). The cost \( \eta_m(t) \) is assumed to be symmetric within the \( m \)-technology sector. Then, \( R_{hm}(t) = \sum e R_{hm}(t) \) and \( \dot{N}_m(t) = \sum e \dot{N}_m(t) \), implying

\[
R_{hm}(t) = \eta_m(t) \cdot \dot{N}_m(t), \quad m \in \{L, H\}. \tag{18}
\]

Concerning the cost of setting up a new variety, \( \eta_m(t) \), we assume that is increasing in both the number of existing varieties, \( N_m(t) \), and the number of new entrants, \( \dot{N}_m(t) \),

\[
\eta_m(t) = \phi \cdot N_m(t)^{1+\sigma} \cdot \dot{N}_m(t)^\gamma, \quad m \in \{L, H\}, \tag{19}
\]

where \( \phi > 0 \) is a fixed (flow) cost, while \( \sigma > 0 \) and \( \gamma > 0 \) relate \( \eta \) with \( N \) and \( \dot{N} \), respectively. Indeed, equation (19) introduces two types of decreasing returns associated to horizontal innovation. Dynamic decreasing returns to scale are modeled by the dependence of \( \eta \) on \( N \) and result from complexity (e.g., Evans, Honkapohja, and Romer, 1998; Barro and Sala-i-Martin, 2004, ch. 6), in the sense that the larger the number of existing varieties, the costlier it is to introduce new varieties. It is noteworthy that the elasticity regulating the horizontal-R&D complexity costs is larger than the one in the vertical-R&D case (i.e., \( 1 + \sigma > 1 \)), in line with what should be expected bearing in mind the distinct nature of the two types of R&D (physical versus immaterial). Static decreasing returns to scale (at the aggregate level) are modeled by the dependence of \( \eta \) on \( \dot{N} \) and mean that one potential entrant exerts an externality on other entrants (e.g., due to congestion effects). This externality is compatible with the previous assumption of constant returns to scale at the firm level (e.g., Arnold, 1998; Jones and Williams, 2000). The dependence of the entry cost on the number of entrants introduces dynamic second-order effects from entry, implying that new varieties are brought to the market gradually, instead of through a lumpy adjustment. This is in line with the stylised facts.
on entry (e.g., Geroski, 1995): entry occurs mostly at small scale since adjustment costs penalise large-scale entry.

Every horizontal innovation results in a new intermediate good whose quality level is drawn randomly from the distribution of existing varieties (e.g., Howitt, 1999). Thus, the expected quality level of the horizontal innovator is

$$\bar{q}_m(t) = \int_0^{N_m(t)} \frac{q_m(\omega_m, t)}{N_m(t)} d\omega_m = \frac{Q_m(t)}{N_m(t)}, \ m \in \{L, H\},$$  \hspace{1cm} (20)

As his/her monopoly power will be also terminated by the arrival of a successful vertical innovator in the future, the benefits from entry are given by

$$V_m(q_m) = \pi_m \cdot \bar{q}_m(t) \int_t^{\infty} P_m(t)^{\frac{1}{2}} \cdot e^{-\int_t^{s} [r(\nu) + I_m(q_m)] d\nu} ds, \ m \in \{L, H\},$$  \hspace{1cm} (21)

where \( \pi_m q_m = \bar{\pi}_m P_m^{\frac{1}{2}} \). The free-entry condition is now \( \dot{N}_m \cdot V(q_m) = R_{hm} \), which simplifies to

$$V_m(q_m) = \frac{\eta_m(t)}{N_m(t)}, \ m \in \{L, H\}.$$  \hspace{1cm} (22)

Substituting (21) into (22) and time-differentiating the resulting expression, yields the no-arbitrage condition facing a horizontal innovator

$$r(t) + I_m(t) = \frac{\bar{\pi}_m(t)}{\eta_m(t)/N_m(t)}, \ m \in \{L, H\}.$$  \hspace{1cm} (23)

### 2.2.3. Intra-sector no-arbitrage conditions

No-arbitrage in the capital market requires that the two types of investment – vertical and horizontal R&D – yield equal rates of return; otherwise, one type of investment dominates the other and a corner solution obtains. Thus, if we equate the effective rate of return \( r(t) + I_m(t) \) for both types of entry, from (15) and (23), we get the intra-sector no-arbitrage conditions

$$\bar{q}_L(t) = \frac{\eta_L(t)}{N_L(t)} = \frac{Q_L(t)}{N_L(t)} = \frac{\eta_L(t)}{\zeta \cdot L^e \cdot N_L(t)}, \ \bar{q}_H(t) = \frac{\eta_H(t)}{N_H(t)} = \frac{Q_H(t)}{N_H(t)} = \frac{\eta_H(t)}{\zeta \cdot H^e \cdot N_H(t)}.$$  \hspace{1cm} (24)

which are a key ingredient of the model. They equate the average cost of horizontal R&D, \( \eta_L(t)/N_L(t) \) (respectively, \( \eta_H(t)/N_H(t) \)), to the average cost of vertical R&D, \( \bar{q}_L(t)\zeta L^e (\bar{q}_H(t)\zeta H^e) \).

On the other hand, bearing in mind (19), (24) can be equivalently recast as

$$\dot{N}_m(t) = x_m(Q_m(t), N_m(t)) \cdot N_m(t), \ m \in \{L, H\},$$  \hspace{1cm} (25)

where
\[ x_L(Q_L, N_L) \equiv \left( \frac{\zeta_L}{\phi} \cdot L^\gamma \right)^{\frac{1}{\gamma}} \cdot Q_L^{\frac{1}{\gamma}} \cdot N_L^{\frac{1+a+1}{\gamma}}, \quad (26) \]

\[ x_H(Q_H, N_H) \equiv \left( \frac{\zeta_H}{\phi} \cdot H^\gamma \right)^{\frac{1}{\gamma}} \cdot Q_H^{\frac{1}{\gamma}} \cdot N_H^{\frac{1+a+1}{\gamma}}, \quad (27) \]

which clarifies the adopted mechanism of entry by explicitly incorporating a channel between vertical innovation and firm dynamics. It shows that the horizontal-entry rate, \( \dot{N}_m/N_m \), depends positively on \( Q_m \) and negatively on \( N_m \): the first effect represents complementarity going from vertical innovation, through the technological-knowledge stock, to the horizontal-entry rate, and the second results from the complexity and the congestion effects in horizontal entry (see (19)).

In a small time interval, the growth rate of average quality is equal to the expected arrival rate of a successful innovation multiplied by the quality shift it introduces: \( \tilde{m}_m - \overline{q}_m = I_m(t)(q^+_m - q_m)/q_m \), where both the innovation rate and the quality shift are industry-independent. Time-differentiating (20), and using (25) yields

\[ \dot{Q}_m(t) = (\Xi \cdot I_m(t) + x_m(Q_m(t), N_m(t)) \cdot Q_m(t), \quad m \in \{ L, H \}, \quad (28) \]

where the quality shift is denoted by \( \Xi \equiv (q^+_m - q_m)/q_m = \lambda^{1-\alpha}/\alpha - 1 \). The vertical innovation rate is endogenous and will be determined as an economy-wide function below. Equation (28) introduces a second dynamic interaction between the two types of entry, in this case between the number of varieties and the growth rate of the aggregate quality index.

### 2.3. Households

The economy is populated by a fixed number of infinitely-lived households who consume and collect income from investments in financial assets (equity) and from labour. Households inelastically supply low-skilled, \( L \), or high-skilled labour, \( H \). Thus, total labour supply, \( L + H \), is exogenous and constant. We assume consumers have perfect foresight concerning the technological change over time and choose the path of final-good aggregate consumption \( \{ C(t), t \geq 0 \} \) to maximise discounted lifetime utility

\[ U = \int_0^\infty \left( \frac{C(t)^{1-\theta} - 1}{1-\theta} \right) e^{-\rho t} dt, \quad (29) \]

where \( \rho > 0 \) is the subjective discount rate and \( \theta > 0 \) is the inverse of the intertemporal elasticity of substitution, subject to the flow budget constraint

\[ \dot{a}(t) = r(t) \cdot a(t) + w_L(t) \cdot L + w_H(t) \cdot H - C(t), \quad (30) \]

where \( a \) denotes households’ real financial assets holdings. The initial level of wealth \( a(0) \) is given and the non-Ponzi games condition \( \lim_{t \to \infty} e^{-\int_0^t r(s)ds} a(t) \geq 0 \) is also imposed. The Euler equation for consumption and the transversality condition are standard,
\[
\frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta} \cdot (r(t) - \rho) 
\]  
\[ \lim_{t \to \infty} e^{-\theta t} \cdot C(t)^{-\theta} \cdot a(t) = 0. \]  

2.4. Macroeconomic aggregation and equilibrium innovation rates

The aggregate financial wealth held by all households is \( a(t) = \sum_{m=L,H} \int_0^t N_m(t) \omega_m(t) d\omega_m \), which, from the arbitrage condition between vertical and horizontal entry (24), yields \( a(t) = \sum_{m=L,H} \eta_m(t) \cdot N_m(t) \). Taking time derivatives and comparing with (30), we get an expression for the aggregate flow budget constraint which is equivalent to the product market equilibrium condition (see Appendix A)

\[
Y_{tot}(t) = C(t) + X_{tot}(t) + R_h(t) + R_v(t) 
\]

where \( R_h = \sum_{m=L,H} R_{hm} \) and \( R_v = \sum_{m=L,H} R_{vm} \). Substituting the expressions for the aggregate outputs (10) and (9), and for total R&D expenditures (17) and (18), we have

\[
\chi_Y \cdot \Gamma(t) = C(t) + \chi_X \cdot \Gamma(t) + \eta_L(t) \cdot \dot{\tilde{N}}_L(t) + \eta_H(t) \cdot \dot{\tilde{N}}_H(t) + \\
+ \zeta \cdot \lambda^{\frac{1}{\alpha}} \left( \mathcal{L}^c \cdot I_L(t) \cdot Q_L(t) + \mathcal{H}^c \cdot I_H(t) \cdot Q_H(t) \right). 
\]

Solving for, e.g., \( I_L \), and using (24) and (25), we get the endogenous vertical-innovation rate at equilibrium in the \( L \)-technology sector

\[
I_L(Q_L, Q_H, N_L, N_H, C, I_H) = \frac{1}{\zeta \cdot \lambda^{\frac{1}{\alpha}} \cdot \mathcal{L}^c} \left[ (\chi_Y - \chi_X) \left( [P_H(Q_H, Q_L)]^{\frac{1}{\alpha}} \cdot \mathcal{H} \cdot \frac{Q_H}{Q_L} + [P_L(Q_H, Q_L)]^{\frac{1}{\alpha}} \cdot \mathcal{L} \right) - \\
- \left( \frac{\mathcal{H}}{\mathcal{L}} \right)^{\epsilon} \frac{Q_H}{Q_L} \cdot I_H - \frac{1}{\lambda^{\frac{1}{\alpha}}} \cdot \left[ \left( \frac{\mathcal{H}}{\mathcal{L}} \right)^{\epsilon} \cdot \frac{Q_H}{Q_L} \cdot x_H(Q_H, N_H) + x_L(Q_L, N_L) \right] \right), 
\]

as a function which is decreasing in consumption, increasing in the number of varieties and related in an ambiguous way with the aggregate quality level in each sector \( m \in \{L, H\} \). Observe, in particular, that \( P_L \) and \( P_H \) are (non-linear) functions of \( Q_H/Q_L \) alone (see (6) and (7)). If we further use (16) to eliminate \( I_H \) from (35), we get \( I_L \equiv I_L(Q_L, Q_H, N_L, N_H, C) \). As functions \( I_m(Q_L, Q_H, N_L, N_H, C) \) can be negative, the relevant innovation rates at the macroeconomic level are

\[
I_m^+(Q_L, Q_H, N_L, N_H, C) = \max \{ I_m(Q_L, Q_H, N_L, N_H, C), 0 \}, m \in \{L, H\}. 
\]

Thus, there is also a complementary effect of horizontal innovation on vertical innovation: if the number of varieties is too low, vertical R&D shuts down.\(^{15}\) From (15), we get the

\(^{15}\)This effect is analysed in more detail in Gil, Brito, and Afonso (2013).
rate of return of capital as \( r(Q_L, Q_H, N_L, N_H, C) = r_{0m} - I_m^+(Q_L, Q_H, N_L, N_H, C) \), where \( r_{0L} \equiv \pi_{0L} \frac{1}{1 - \alpha_L} \). As one can see below in Section 3 and illustrated in Appendix B, these conditions are met by our numerical simulations.

2.5. The dynamic general equilibrium

The dynamic general equilibrium is defined by the paths of allocations and price distributions \( \{X_m(\omega_m, t), P_m(\omega_m, t)\}, \omega_m \in [0, N_m(t)]_{t \geq 0} \) and aggregate number of firms, quality indices and vertical-innovation rates \( \{N_m(t), Q_m(t), I_m(t)\}_{t \geq 0} \) for sectors \( m \in \{L, H\} \), and by the aggregate paths \( (C(t), r(t))_{t \geq 0} \), such that: (i) consumers, final-goods firms and intermediate-goods firms solve their problems; (ii) free-entry and no-arbitrage conditions are met; and (iii) markets clear. Total supplies of high- and low-skilled labour are exogenous. We focus on the region of the state space where both \( I_m(\cdot) > 0, m \in \{L, H\} \), such that the equilibrium paths can be obtained from the system

\[
\dot{C} = \frac{1}{\vartheta} \cdot (r_{0m} - I_m^+(Q_L, Q_H, N_L, N_H, C) - \rho) \cdot C \\
Q_m = (I_m^+(Q_L, Q_H, N_L, N_H, C) \cdot \Xi + x_m(Q_m, N_m)) \cdot Q_m \\
N_m = x_m(Q_m, N_m) \cdot N_m
\]

(37)-(39)

given \( Q_m(0) \) and \( N_m(0) \), and the transversality condition (32), which may be re-written as

\[
\lim_{t \to \infty} e^{-\rho t} \cdot C(t)^{-\vartheta} \cdot \zeta \cdot \left( L^c \cdot Q_L(t) + H^c \cdot Q_H(t) \right) = 0.
\]

(40)

2.6. The balanced-growth path

As the functions in system (37)-(39) are homogeneous, a BGP exists only if: (i) the asymptotic growth rates of consumption and of the quality indices are constant and equal to the economic growth rate, \( g_C = g_{Q_L} = g_{Q_H} = g \); (ii) the asymptotic growth rates of the number of varieties are constant and equal, \( g_{N_L} = g_{N_H} \); (iii) the vertical-innovation rates and the final-goods price indices are asymptotically trendless, \( g_I_L = g_I_H = g_P_L = g_P_H = 0 \); and (iv) the asymptotic growth rates of the quality indices and the number of varieties are monotonously related, \( g_{Q_L}/g_{N_L} = g_{Q_H}/g_{N_H} = (\sigma + \gamma + 1) \cdot g_{N_m} \neq 0 \), \( m \in \{L, H\} \).

Observe, from 25, that \( x_m = g_{N_m} \) is always positive if \( N_m > 0 \).

It will be convenient to recast system (37)-(39), by using variables \( x_m \) as in (26)-(27) and variables \( z_L \equiv C/Q_L \) and \( Q \equiv Q_H/Q_L \), into an equivalent system in detrended variables. We then get

\[
\dot{x}_L = \left[ \Xi \cdot I_L - \left( \frac{\sigma + \gamma}{\gamma} \right) \cdot x_L \right] \cdot x_L,
\]

(41)

\[\text{As one can see below in Section 3 and illustrated in Appendix B, these conditions are met by our numerical simulations.}\]
\[ \dot{z}_L = \left[ \frac{1}{\theta} \cdot (r_{0L} - \rho) - \left( \frac{1}{\theta} + \Xi \right) \cdot I_L - x_L \right] \cdot z_L, \quad (42) \]

\[ \dot{x}_H = \left[ \frac{\Xi}{\gamma} \cdot I_L - \left( \frac{\sigma + \gamma}{\gamma} \right) \cdot x_H + \frac{\pi_0}{\zeta} \cdot \left( H^{1-\epsilon} \cdot P_H^{1\epsilon} - L^{1-\epsilon} \cdot P_L^{1\epsilon} \right) \right] \cdot x_H, \quad (43) \]

\[ \dot{Q} = \left[ \Xi \cdot \frac{\pi_0}{\zeta} \cdot \left( H^{1-\epsilon} \cdot P_H^{1\epsilon} - L^{1-\epsilon} \cdot P_L^{1\epsilon} \right) + x_H - x_L \right] \cdot Q, \quad (44) \]

where \( I_L \equiv I_L(Q, x_L, x_H, z_L) = I_L(Q_L, Q_H, N_L, N_H, C), \) \( I_H \equiv I_H(Q, x_L, x_H, z_L) = I_H(Q_L, Q_H, N_L, N_H, C), \) \( P_L \equiv P_L(Q) = P_L(Q_L, Q_H), \) and \( P_H \equiv P_H(Q) = P_L(Q_L, Q_H). \)

These equations, together with the transversality condition (40) and the initial conditions \( x_L(0), x_H(0) \) and \( Q(0), \) describe the transitional dynamics and the BGP, by jointly determining \( x_L(t), z_L(t), x_H(t) \) and \( Q(t). \) Then, we can determine the level variables \( N_m(t), C(t) \) and \( Q_L(t) \) (respectively, \( Q_H(t) \)), for a given \( Q_H(t) (Q_L(t)). \)

The households transversality condition (40) can also be related to the detrended variables,

\[ \lim_{t \to \infty} e^{-\alpha t} \cdot z_L(t)^{-\theta} \cdot \zeta \cdot (L^e + H^e \cdot Q(t)) \cdot Q_L(t)^{1-\theta} = 0 \quad (45) \]

where \( z_L \) and \( Q \) are stationary along the BGP, as shown above. Let \( Q_L = \hat{q}_Le^{\theta t}, \) where \( \hat{q}_L \) denotes detrended \( Q_L \) (thus stationary along the BGP), and substitute in (45), to see that the transversality condition implies \( \rho \geq (1 - \theta)g; \) using the Euler equation, \( g = (r - \rho) / \theta, \) the latter condition can be written alternatively as \( r > g. \) As it happens, this condition also guarantees that attainable utility is bounded, i.e., the integral (29) converges.

**Proposition 1.** The interior steady state exists and is unique.

By equating \( \dot{x}_m = \dot{z}_L = \dot{Q} = 0, m \in \{ L, H \}, \) and under the condition \( \tilde{r}_{0m} - \rho > 0, \) it is straightforward to show that there is just one interior steady state,

\[ \tilde{Q} = \left( \frac{H}{L} \right)^{1-2\epsilon}, \quad (46) \]

\[ \tilde{x}_L = \tilde{x}_H = \frac{\Xi}{\theta} (\tilde{r}_{0L} - \rho) \quad (47) \]

\[ \tilde{z}_L = (\chi_Y - \chi_X) \left( \tilde{P}_H^{1\epsilon} H \tilde{Q} + \tilde{P}_L^{1\epsilon} L \right) - \left( \zeta \lambda^{1-\alpha} \tilde{I}_L + \zeta \tilde{x}_L \right) \left( H^e \tilde{Q} + L^e \right), \quad (48) \]

where \( \tilde{I}_L = \tilde{I}_H = \left( \frac{\sigma + \gamma}{\gamma} \right) \tilde{x}_L = \left( \frac{\sigma + \gamma}{\Xi} \right) \tilde{x}_H; \) \( \tilde{r}_{0L} \equiv \frac{\pi_0}{\zeta} L^{1-\epsilon} \tilde{P}_L^\epsilon = \tilde{r}_{0H} \equiv \frac{\pi_0}{\zeta} H^{1-\epsilon} \tilde{P}_H^\epsilon; \)

\( \tilde{P}_L = e^{-\alpha} \left[ 1 + (H/L)^{1-\epsilon} \right], \) and \( \tilde{P}_H = e^{-\alpha} \left[ 1 + (H/L)^{\epsilon-1} \right]. \) Equations (46)-(48) represent a steady-state equilibrium with balanced growth in the usual sense, such that the endogenous growth rates are positive,

\[ \tilde{g}_N_L \equiv \tilde{g}_N_H \equiv \tilde{x}_L = \tilde{x}_H > 0, \quad (49) \]
Thus, our model predicts, under a sufficiently productive technology, a BGP with constant positive growth rates, \( g \) and \( g_{N_m} \), where the former exceeds the latter by the growth of intermediate-good quality; to verify this, just replace (25) in (28), and solve to get \( \dot{Q}_m/Q_m - \dot{N}_m/N_m = I_m \cdot \Xi, m \in \{L, H\} \), which is positive since \( I_m > 0 \). This implies that the economic growth rate equals the growth rate of the number of varieties plus the growth rate of intermediate-good quality, in line with the well-known view that industrial growth proceeds both along an intensive and an extensive margin. However, reflecting the distinct nature of vertical and horizontal innovation (immaterial versus physical) and the consequent asymmetry in terms of R&D complexity costs (see (11) and (19)), variety expansion is sustained by the endogenous quality upgrade, as the expected growth of intermediate-good quality due to vertical R&D makes it attractive, in terms of intertemporal profits, for potential entrants to always put up an entry cost, in spite of its increase with \( N_m \). Thus, vertical innovation arises as the ultimate growth engine, in the sense that it sustains both variety expansion and aggregate output growth.

The level variables are \( \tilde{C}, \tilde{N}_m, \) and \( \tilde{Q}_m, m \in \{L, H\} \), where \( \tilde{Q}_L \) is undetermined and

\[
\tilde{C} = \tilde{z} \tilde{Q}_L, \tag{51}
\]

\[
\tilde{N}_L = \left( \frac{\zeta}{\phi} L^e \right) \sigma+\gamma+1 (\tilde{x}_L) \frac{1}{\sigma+\gamma+1} (\tilde{Q}_L) \frac{1}{\sigma+\gamma+1}, \tag{52}
\]

\[
\tilde{N}_H = \left( \frac{\zeta}{\phi} H^e \right) \sigma+\gamma+1 (\tilde{x}_H) \frac{1}{\sigma+\gamma+1} \left[ (\frac{H}{L})^{1-2\epsilon} \tilde{Q}_L \right] \frac{1}{\sigma+\gamma+1}. \tag{53}
\]

From the expressions for \( X_L \) and \( X_H \) (see (9)) and for \( N_L \) and \( N_H \) above, combined with (46) and (47), we derive the expressions for relative intermediate-good production and the relative number of firms (i.e., \( H \)-vis-à-vis \( L \)-technology sector),

\[
\tilde{X} \equiv \left( \frac{\tilde{X}_H}{\tilde{X}_L} \right) = \left( \frac{H}{L} \right)^{1-\epsilon}, \tag{54}
\]

\[
\tilde{N} \equiv \left( \frac{\tilde{N}_H}{\tilde{N}_L} \right) = \left( \frac{H}{L} \right)^{\frac{1-\epsilon}{\sigma+\gamma+1}}. \tag{55}
\]

Since firm size may be measured as production (or sales) per firm, then, by considering (54) and (55), we get the relative firm size,

\[
\left( \frac{\tilde{X}}{\tilde{N}} \right) = \left( \frac{H}{L} \right)^{\frac{(1-\epsilon)(\sigma+\gamma)}{\sigma+\gamma+1}}. \tag{56}
\]

**Proposition 2.** The interior steady state is locally saddle-path stable.
Linearising in a neighbourhood of the interior steady state \((\tilde{x}_L, \tilde{z}_L, \tilde{x}_H, \tilde{Q})\), the dynamics may be approximated by the following fourth-order system obtained from (41)-(44),

\[
\begin{pmatrix}
\dot{\tilde{x}}_L \\
\dot{\tilde{z}}_L \\
\dot{\tilde{x}}_H \\
\dot{\tilde{Q}}
\end{pmatrix} =
\begin{pmatrix}
a_{11} \tilde{x}_L & a_{12} \tilde{x}_L & a_{13} \tilde{x}_L & \frac{\Xi}{7} \left( \frac{\partial I}{\partial Q} \right) \tilde{x}_L \\
-\frac{1}{\xi} \tilde{z}_L & a_{21} \tilde{z}_L & a_{22} \tilde{z}_L & a_{23} \tilde{z}_L & a_{24} \tilde{z}_L \\
-\frac{1}{\xi} \tilde{x}_H & a_{31} \tilde{x}_H & a_{32} \tilde{x}_H & a_{33} \tilde{x}_H & a_{34} \tilde{x}_H \\
0 & -\Xi S L_1 & 0 & -\Xi S L_2 & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{x}_L \\
\tilde{z}_L \\
\tilde{x}_H \\
\tilde{Q}
\end{pmatrix},
\tag{57}
\]

given the initial conditions \(x_L(0), x_H(0)\) and \(Q(0)\) and the transversality condition (45). The Jacobian matrix \(J(\tilde{x}_L, \tilde{x}_H, \tilde{z}_L, \tilde{Q})\), in (57), is evaluated at the steady state, where we define

\[
\begin{align*}
a_{11} & \equiv -\frac{3}{7} \left( \frac{3}{\xi} + \frac{2}{1-\xi} \right) S_0 - \frac{\sigma + \gamma}{7} \xi; & a_{12} & \equiv -\frac{1}{7 \xi} \left( \frac{3}{\xi} + \frac{2}{1-\xi} \right) S_0; & a_{13} & \equiv -\frac{3}{7 \xi} \left( \frac{3}{\xi} + \frac{2}{1-\xi} \right) S_0 (\frac{\tilde{Q}}{2})^{1-\epsilon}; \\
a_{21} & \equiv \left( \frac{1}{\tilde{Q}} + \frac{3}{\xi} \right) \frac{1}{\xi} S_0 - 1; & a_{22} & \equiv \left( \frac{1}{\tilde{Q}} + \frac{3}{\xi} \right) \frac{1}{\xi} S_0; & a_{23} & \equiv \left( \frac{1}{\tilde{Q}} + \frac{3}{\xi} \right) \frac{1}{\xi} S_0; \\
a_{24} & \equiv \frac{30}{\xi} \frac{1}{\tilde{Q}} \Gamma^{1-\epsilon} \left( \frac{\tilde{Q}}{2} \right)^{1-\epsilon} - \left( \frac{1}{\tilde{Q}} + \frac{3}{\xi} \right) \frac{1}{\xi} \frac{1}{\gamma} \frac{1}{\xi} S_0; \\
a_{31} & \equiv a_{11} + \frac{\sigma + \gamma}{7} \xi; & a_{32} & \equiv a_{12}; & a_{33} & \equiv a_{13} - \frac{\sigma + \gamma}{7} \xi; & a_{34} & \equiv a_{14} - \frac{3}{\xi} S L_1;
\end{align*}
\]

with

\[
S_0 \equiv 1/ \left[ 1 + \left( \frac{\tilde{Q}}{2} \right)^{1-\epsilon} \right]; S L_1 \equiv \frac{30}{\xi} \frac{1}{\tilde{Q}} \Gamma^{1-\epsilon} \frac{1}{Q S L_0};
\]

\[
\left( \frac{\partial I}{\partial Q} \right) = \left[ S L_1 - \left( \frac{1}{\xi} + \frac{\sigma + \gamma}{7} \right) \frac{1}{\xi} S L_2 \right] \left( \frac{\tilde{Q}}{2} \right)^{1-\epsilon} S_0 + \frac{1}{\xi} \frac{1}{\gamma} \frac{1}{\xi} S_0 \left( \chi Y - \chi X \right) \frac{1}{\tilde{Q}}.
\]

Since there are three predetermined variables, \(x_L, x_H\) and \(Q\), and one jump variable, \(z_L\), saddle-path stability of the interior equilibrium \((\tilde{x}_L, \tilde{x}_H, \tilde{z}_L, \tilde{Q})\) requires that \(J(\tilde{x}_L, \tilde{x}_H, \tilde{z}_L, \tilde{Q})\) has three eigenvalues with a negative real part and one with a positive real part, hence implying \(\text{det}(J(\tilde{x}_L, \tilde{x}_H, \tilde{z}_L, \tilde{Q})) < 0\). After some algebra, we get

\[
\text{sgn} \left\{ \text{det}(J(\tilde{x}_L, \tilde{x}_H, \tilde{z}_L, \tilde{Q})) \right\} = \text{sgn} \left\{ -\Xi S L_1 + \left( \frac{\sigma + \gamma}{7} \right) a_{41} + a_{12} (a_{23} + a_{21}) - a_{22} a_{11} + \Xi S L_1 a_{13} a_{22} - \left( \frac{\sigma + \gamma}{7} \right) a_{44} a_{22} a_{13} \right\}.
\]

Given that \(a_{11} < 0, a_{12} < 0, a_{13} < 0, a_{22} > 0, a_{23} > 0, a_{44} < 0\), then a sufficient condition for \(\text{det}(J(\tilde{x}_L, \tilde{x}_H, \tilde{z}_L, \tilde{Q})) < 0\) is \(a_{12} (a_{23} + a_{21}) - a_{22} a_{11} < 0\). However, since \(\text{det}(J(\tilde{x}_L, \tilde{x}_H, \tilde{z}_L, \tilde{Q})) < 0\) is compatible with both one and three eigenvalues with negative real part, further, more complicated, conditions must be satisfied so that saddle-path stability applies. These conditions are met by our numerical simulations for our baseline parameter values, but also over a wide range of parameter sets.

Finally, it is noteworthy that, since the dimension of the stable manifold is larger than unity (it is three-dimensional), there are multiple independent sources of stability in the dynamic system, but which interact between themselves. Thus, non-monotonic

\[\text{Since the characteristic equation for the linearised system (57) is of the form } \beta^4 - b_1 \beta^3 + b_2 \beta^2 - b_3 \beta + b_4,\]

where the coefficients \(b_k, k = 1, \ldots, 4\), equal the sum of all \(k\)th-order principal minors of \(J\), these conditions rely on the solution for the quartic equation (see, e.g., Abramowitz and Stegun, 1972; King, 1996) and hence are particularly hard to check analytically, except in particular cases.

\[\text{These numerical results are available from the authors upon request.}\]
trajectories can emerge in the pre-determined variables along transition even in the case of a linearised dynamic system (see, e.g., Eicher and Turnovsky, 2001, whose endogenous growth model features a two-dimensional stable manifold).

3. Industry and aggregate dynamics

3.1. Comparative dynamics

This section focuses on the change of the industry structure (high- versus low-tech sectors) within manufacturing over time and on its compatibility with stable aggregate variables, namely the economic growth rate and the real interest rate. To that end, we explore the transitional dynamics results of the model triggered by an unanticipated one-off shock in the proportion of high-skilled labour. Global dynamics, as opposed to local dynamics, allows us to carry out a comparative dynamics exercise without restricting the analysis to an arbitrarily close neighbourhood of the steady state and, thus, to arbitrarily small shifts in the parameters and the exogenous variables. As shown in the previous section, the dynamic system in detrended variables is four dimensional, with three pre-determined endogenous variables, and is highly non-linear. Therefore, we resort to numerical methods to study its global dynamics.

We begin by assigning values to the different parameters and exogenous variables, which decide on the starting and the arriving point of the simulation (initial and final steady state), and by defining the variables of interest for the analysis. The model is then solved by numerical integration for different scenarios, using a finite difference method implementing the three-stage Lobatto IIIa formula. The code performs a mesh selection and error control based on the residual of the continuous solution and is provided through the MatLab 7 function bvp4c.\(^{19}\)

Bearing in mind the derivations carried out in Section 2, the time-path solutions of the three predetermined variables, \(x_L, x_H\) and \(Q\), and the jump variable, \(z_L\), allow us to assess the industry dynamics, measured by the:

- Relative number of firms (the ratio of the number of firms in the \(H\)- to the \(L\)-technology sector),

\[
N(t) = \left( \frac{H}{L} \right)^{\frac{\gamma}{\sigma + \gamma + 1}} \left( \frac{x_H(t)}{x_L(t)} \right)^{-\left(\frac{\gamma}{\sigma + \gamma + 1}\right)} Q(t)^{\frac{1}{\sigma + \gamma + 1}}; \quad \text{(58)}
\]

- Relative production (the ratio of production in the \(H\)- to the \(L\)-technology sector), that were

\[
X(t) = \left( \frac{H}{L} \right)^{\frac{1}{2}} Q(t)^{\frac{3}{2}}; \quad \text{(59)}
\]

\(^{19}\)As an alternative numerical procedure, we also used the “Forward Shoot 1D” algorithm by Atolia and Buffie (2009), which is a Mathematica software implementation. In the case of our dynamic system, which has three pre-determined endogenous variables, this numerical method yielded similar results to the MatLab built-in algorithm but with a prohibitive computational time, especially when several executions were to be made.

21
• Sectoral growth rate in the \( L \)-technology sector,
\[
g_{Q_H}(t) = I_L(t) \cdot \Xi + x_L(t); \tag{60}
\]
• Sectoral growth rate in the \( H \)-technology sector,
\[
g_{Q_L}(t) = I_H(t) \cdot \Xi + x_H(t). \tag{61}
\]

At the aggregate level, the dynamics is analysed by computing the:

• Economic growth rate,
\[
g(t) = \frac{L^{\frac{1}{2}} \cdot g_{Q_L}(t) + (Q(t) \cdot H)^{\frac{1}{4}} \cdot g_{Q_H}(t)}{L^{\frac{1}{2}} + (Q(t) \cdot H)^{\frac{1}{4}}}; \tag{62}
\]

• Real interest rate,
\[
r(t) = \frac{\pi_0}{\zeta} \cdot L^{1-\epsilon} \cdot P_L(t)^{\frac{1}{2}} - I_L(t) = \frac{\pi_0}{\zeta} \cdot H^{1-\epsilon} \cdot P_H(t)^{\frac{1}{2}} - I_H(t). \tag{63}
\]

The effects of a shock in the relative supply of skills, \( H/L \), on the variables of interest are then studied under three different scenarios for the complexity cost parameter, \( \epsilon \) (and thus the degree of scale effects on industrial growth, \( 1 - \epsilon \)). The three scenarios feature, relatively to the baseline case, a rise in \( H/L \) by considering a jump in high-skilled labour, \( H \), from 0.3 to 0.5, while the low-skilled labour, \( L \), is normalised to unity. This then implies that the initial and the new steady state are characterised by, respectively, \( H/L = 0.3 \) and \( H/L = 0.5 \). These correspond to the average value of the ratio of high- to low-skilled labour (college versus non-college graduates) for, respectively, the decades of the 80’s and the 90’s, across a number of developed countries, as reported by Acemoglu (2003).\(^{20}\)

In Scenario 1, we focus on \( \epsilon = 0 \) or values of \( \epsilon \) near zero, that is, following, e.g., Acemoglu (1998) and Acemoglu and Zilibotti (2001), market-scale effects prevail. Scenario 2 is characterised by \( \epsilon = 1 \) or values of \( \epsilon \) near unity, in which case the market-scale effects are (totally or almost totally) removed and the price-channel effects prevail, in line with Jones (1995a) and others. Finally, in Scenario 3, we let \( \epsilon = 0.5 \), meaning that market-size-channel and price-channel effects offset each other exactly, such that the technological-knowledge bias, \( Q \), is independent of the relative supply of skills, \( H/L \), on the BGP.

As for the remaining parameters of the model, we define the following set of baseline values: \( A = 1; \alpha = 0.6; \lambda = 2.5; \zeta = 0.6; \rho = 0.02; \theta = 1.5; \sigma = 1.2; \gamma = 1.2; l = 1.0; h = 1.3.\(^{21}\) Given that, along the BGP, we have \( g_{Q_m} - g_{N_m} = (\sigma + \gamma)g_{N_m} \), we let

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\(^{20}\) We resort to this source since the Eurostat online database on educational attainment of employed population only covers the 1995-2007 period.

\(^{21}\) The value of the discount rate, \( \rho \), implies that each period in our model represents a year.
Scenario 1 \((\epsilon = 0)\) 0.55
Scenario 2 \((\epsilon = 1)\) 0.67 0.025 0.021 0.057
Scenario 3 \((\epsilon = 0.5)\) 0.60

Table 1: Calibration of the vertical-R&D flow fixed cost, \(\zeta\), under three scenarios for \(\epsilon\), in order to match the cross-country average of the per capita GDP growth rate over the period 1995-2007, for a sample of 14 European countries in the Eurostat on-line database (available at \http://epp.eurostat.ec.europa.eu\).

\(\sigma + \gamma = 2.4\) to match the ratio between the growth rate of the average firm size and the growth rate of the number of firms found in cross-country data in the period 1995-2007, while the values for \(l\) and \(h\) are in line with Afonso and Thompson (2011). Since they have no impact on the growth rates, \(\phi\) and \(A\) were normalised to unity, while the values for \(\theta\), \(\rho\), \(\lambda\) and \(\alpha\) were set in line with the standard literature (see, e.g., Barro and Sala-i-Martin, 2004). The value of the remaining parameter, \(\zeta\), was chosen in order to calibrate the after-shock BGP economic growth rate, \(g\), around 2.5 percent/year (see Table 1), matching the cross-country average of the per capita GDP growth rate over the period 1995-2007.22 Then, the implied value for the Poisson rate, \(I\), is around 2.1 percent/year; this means that the model predicts an average lifetime of a design of 47.6 years, which is within the range of values considered in the empirical literature (see Strulik, 2007). Moreover, the implied value for the real interest rate, \(r\), is 5.7 percent, broadly in line with the empirical value for the long-run average real return on the stock market, and which should be taken as the equilibrium rate of return to R&D (see, e.g., Jones and Williams, 2000). Nonetheless, extensive sensitivity analysis has shown that the results presented hereafter are robust, in qualitative terms, to changes in the underlying parameters.

In what follows, we are interested in analysing both the long-run effects (shift in the BGP values) and its decomposition into short-run and medium-run (transitional dynamics) effects of a unanticipated one-off increase in the relative supply of skills, \(H/L\). In particular, we consider an increase in the amount of high-skilled labour, \(H\), with the low-skilled labour, \(L\), remaining constant through time.23 As we will see, the degree

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22The source of the referred to cross-country data is the Eurostat on-line database (link at \http://epp.eurostat.ec.europa.eu\). The sample of 14 countries used to compute the cross-section average is the same as the one used in Figure 1, Section 1. See also fn.2.

23Available data suggests that increases in \(H\) have been larger in modulus than the decreases in \(L\) over time: the annual average variation of college (the usual proxy for high-skilled labour) and non-college graduates (the proxy for the low-skilled) among persons employed in manufacturing was, respectively, 2.7 and -1.0 percent, computed as the average of 20 European countries for the 1995-2008 period. The data is from the Eurostat on-line database on Science, Technology and Innovation (available at \http://epp.eurostat.ec.europa.eu\).
of scale effects, $1 - \epsilon$, is a key, albeit indirect, determinant of the characteristics of transitional dynamics, by influencing simultaneously the short- and the long-run response to the shock.

**Scenario 1 - “Market-size-channel effect prevails” (small $\epsilon$, Figure 2)**

**Industry dynamics: short-run effect.** The increase in $H$ generates an increase in resources in terms of the final good (see (10)) available for R&D. However, the allocation of resources is nonbalanced between sectors. The direct strong positive impact on the profitability of the production of intermediate goods in the $H$-technology sector (see (8)) more than compensates for the decrease in the price index, $P_H$, due to the fall in the marginal productivity of labour of that sector; then, an increase in the vertical-innovation rate $I_H$ occurs due to the predominance of the market-size channel. Moreover, given that $L$ is constant, profits in the $H$-technology sector increase more than in the $L$-technology sector. The diversion of resources from the latter to the former sector induces a fall in $I_L$, although only slightly because of the countervailing effect of the upward jump in the price index, $P_L$. As a result, the sectoral growth rate in the $H$-technology sector, $g_{Q_H}$ jumps upwards, while the growth rate in the $L$-technology sector, $g_{Q_L}$, experiences a small shift downwards.\(^\text{24}\)

**Industry dynamics: medium-run effect.** After the initial jump, $g_{Q_H}$ takes a downward path, while $g_{Q_L}$ follows an upward path; the former reflects the behaviour of the intensive margin (the vertical innovation rate, $I_H$, falls over transition) which more than compensates for the extensive margin (the growth rate of the number of varieties, $x_H$, increases); in contrast, the increase in $g_{Q_L}$ reflects the behaviour of both the intensive and the extensive margin ($I_L$ and $x_L$ increase). After the initial level effect, we have $I_H > I_L$, whereas the time-paths of $I_H$ and $I_L$ respond to a feedback effect: $I_H$ and $I_L$ are commanded by the dynamics of the price indices – $P_H$ decreases and $P_L$ increases towards the new steady state –, which, in turn, reflects the increase in the technological-knowledge bias, $Q$; the bias rises, at a decreasing rate, due to the difference in profitability between the $H$- and the $L$-technology sector, and hence between $I_H$ and $I_L$, induced by the initial jump in $H$. In turn, $x_H$ and $x_L$ rise due to the increase in the sectoral technological-knowledge, $Q_H$ and $Q_L$ (given $I_H > 0$ and $I_L > 0$), reflecting the complementarity between the horizontal-entry rate and the technological-knowledge stock (see (25)); however, the fact that $I_H > I_L$ means that the costs pertaining to horizontal entry are only slightly compensated for in the $L$-technology sector at the beginning of the transition path (see (41) and (43)), while the opposite occurs in the other sector, therefore explaining the different shape of the time-paths of $x_H$ and $x_L$ (concave and convex, respectively). Since $x_H > x_L$ throughout transition, the relative number of firms, $N$, increases. However, the $H$-technology sector experiences an acceleration in terms of the extensive margin that exceeds the one in the $L$-technology sector, as explained earlier; as a result, the congestion effects in horizontal R&D reduce the velocity of convergence of

\(^{24}\)Notice that, since $x_L$, $x_H$ and $Q$ are pre-determined variables in the dynamic system (41)-(44), they do not experience any short-term response to the exogenous shock.
In contrast, the absence of congestion effects in vertical R&D determines a faster increase in relative production, \( X \), commanded by \( Q \) (see (59)),\(^{25}\) and thus also a rise in the relative firm size, \( X/N \).\(^{26}\)

**Industry dynamics: long-run effect.** Both \( g_{Q_H} \) and \( g_{Q_L} \) settle down at a level that is higher than the pre-shock BGP level, reflecting the net positive scale effect (market-size effect) associated to the exogenous shock. Overall, the model predicts that the short-run positive scale effect in the economic growth rate overshoots the long-run positive scale effect in the \( H \)-technology sector, while, in the \( L \)-technology sector, the negative short-run scale effect is more than compensated by the long-run positive scale effect. The relative number of firms, relative production and relative firm size all increase relatively to the pre-shock BGP level.

**Aggregate dynamics** The economic growth rate, \( g \), and the real interest rate, \( r \), experience only a very slight increase along the transition path; thus, the long-run effect of an increase in \( H \) results almost entirely from the short-run response to the exogenous shock.

[Figure 2 goes about here]

**Scenario 2 - “Price-channel effect prevails” (large \( \epsilon \), Figure 3)**

**Industry dynamics: short-run effect** By removing the scale effects, the chain of effects is induced by the price channel, by which there are stronger incentives to improve technologies when the goods that they produce command higher prices. Hence, the direct positive impact of the increase in \( H \) on the profitability of the production of intermediate goods in the \( H \)-technology sector is now more than compensated by the decrease in the price index, \( P_H \); then, a decrease in the vertical-innovation rate \( I_H \) occurs due to the predominance of the price channel. Consequently, a diversion of resources arises from the \( H \)- to the \( L \)-technology sector, inducing an increase in \( I_L \). As a result, the sectoral growth rate in the \( H \)-technology sector, \( g_{Q_H} \) jumps downwards, while the growth rate in the \( L \)-technology sector, \( g_{Q_L} \), experiences a shift upwards.

**Industry dynamics: medium-run effect** After the initial jump, \( g_{Q_H} \) takes an upward path, while \( g_{Q_L} \) follows a downward path. In order to decompose this behaviour in terms of intensive and extensive margin, it is convenient to consider two separate cases, one for \( \epsilon \in (0.5; \bar{\epsilon}) \) and the other for \( \epsilon \in (\bar{\epsilon}; 1] \), where \( \bar{\epsilon} \in (0.5; 1) \) depends on the values of the other parameters.

\(^{25}\)Observe that \( Q \) has also a direct effect on \( N \) (see (58)), but it is dampened by the complexity and congestion effects associated to horizontal R&D and which are regulated by parameters \( \sigma \) and \( \gamma \).

\(^{26}\)Eventually, \( X/N \) will take a slight fall as the economy gets closer to the new BGP because, since the speed of convergence of \( X \) is larger than that of \( N \) (see Figure 5, below), the former will stop increasing before the latter.
Figure 2: Scenario 1: short-run effects and transitional dynamics of the aggregate and industry-level variables when $H$ increases from 0.3 to 0.5. $\epsilon = 0$. Values for $X$ and $N$ are adjusted by their pre-shock initial values, $X(0) = X_0$ and $N(0) = N_0$, to facilitate the comparison between the time variation of those two variables.

(a) With $\epsilon$ up to $\bar{\epsilon}$, the reduction of the sectoral growth rate in the L-technology sector reflects the behaviour of the intensive margin (i.e., the fall in vertical innovation rate, $I_L$), which more than compensates the extensive margin (the growth rate of the number of varieties, $x_L$, increases over most part of the transition path); in contrast, the acceleration of activity in the H-technology sector reflects the behaviour of both the intensive and the extensive margin ($I_H$ increases monotonically along the transition path, while $x_H$ increases over most part of the transition path). After the initial level effect, we have $I_H < I_L$, with $I_H$ and $I_L$ are commanded by, respectively, the increase in $P_H$ and the decrease in $P_L$ towards the new steady state, which, in turn, reflect the decrease in the technological-knowledge bias, $Q$; the bias falls, at a decreasing rate, due to the difference in profitability between the H- and the L-technology sector, and hence between $I_H$ and $I_L$, induced by the initial jump in $H$. In turn, $x_H$ and $x_L$ rise due to the increase in the sectoral technological-knowledge, $Q_H$ and $Q_L$, given $I_H > 0$ and $I_L > 0$; however, the fact that $I_H < I_L$ means that the costs pertaining to horizontal entry are only slightly compensated for in the H-technology sector at the beginning of the transition path, while the opposite occurs in the other sector, which explains the distinct shape of the time-paths of $x_H$ and $x_L$ (the shapes are symmetric to the ones in Scenario 1). Since $x_H < x_L$ all over transition, the relative number of firms, $N$, 

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decreases. However, the L-technology sector experiences an acceleration in terms of the extensive margin that exceeds the one in the H-technology sector, as already explained; hence, the congestion effect pertaining to horizontal R&D reduces the velocity at which $N$ is falling. Benefiting from the absence of congestion effects in vertical R&D, relative production, $X$, takes a faster fall commanded by $Q$, and thus inducing a decrease in the relative firm size, $X/N$.\(^{27}\)

(b) When $\epsilon > \bar{\epsilon}$, $x_H$ and $x_L$ display marked non-monotonic time paths, the former being convex and the latter being concave. As already explained, after the initial level effect, we have $I_H < I_L$. However, as the price channel gets stronger (i.e., $\epsilon$ increases towards unity), the downward jump in $I_H$ becomes larger, such that eventually the vertical-innovation rate is not able to compensate for the costs pertaining to horizontal entry at the beginning of the transition path. Under this scenario, the horizontal entry rate $x_H$ will start the transitional dynamics by following a downward path, but since $I_H$ increases monotonically over transition, the latter will eventually become large enough to overturn the costs effect; from that point on, $x_H$ will take an upward path towards the new steady state.\(^{28}\) In the L-technology sector, an opposite behaviour will occur. Thus, in both sectors, the transition process begins propelled by the intensive margin, although partially countervailed by the extensive margin, but eventually the convergence to the long-run equilibrium is carried out at the expense of both margins. The relative number of firms, relative production and relative firm size are characterised by a behaviour that is similar to the one in (a).

**Industry dynamics: long-run effect** The effect on the industrial growth rates, relative production and the relative number of firms is very small (if $\epsilon$ is near unity) or non-existent (if $\epsilon = 1$).

**Aggregate dynamics** The growth rate and the real interest rate remain approximately constant in response to the shock in $H$, exhibiting time-paths that are (slightly) non-monotonic (in the case of the real interest rate) and very flat over transition, since scale effects are totally (or almost totally) removed from the model.

[Figure 3 goes about here]

---

\(^{27}\)Eventually, $X/N$ will increase slightly as the economy approaches the new BGP because $X$ converges at a higher speed than $N$ (see Figure 5, below).

\(^{28}\)Notice that when the market-size channel prevails, as in Scenario 1, the fall in $I_L$ is only slight because of the countervailing effect of the upward jump in the price index, $P_L$. Thus, a non-monotonic behaviour of $x_L$ does not occur or is very mild.
Scenario 3 - “Balanced market-size-channel and price-channel effects” (ε = 0.5, Figure 4)

Industry dynamics: short-run effect For intermediate values of ε, the market-size and the price channel are in action with similar strength, which implies that the incentives for vertical R&D arising from the shock in $H$ tend to be shared roughly equally between the $L$- and the $H$-technology sector. Overall, this means that more resources become available for a simultaneous, but relatively small, increase in the vertical-innovation rates, $I_L$ and $I_H$, and hence in the sectoral growth rates, $g_{Q_L}$ and $g_{Q_H}$.

Industry dynamics: medium-run effect The endogenous variables experience only a slight (or no) change along the transition path in both sectors, reflecting the balance between the market-size and the price channel; in particular, this balance determines that the technological-knowledge bias, $Q$, is unresponsive to changes in the proportion of high-skilled labour. Both $g_{Q_L}$ and $g_{Q_H}$ then follow upward paths along the transition to the new steady state, with the acceleration of economic activity now being commanded by the extensive margin in both sectors, since $x_H$ and $x_L$ increase over transition. This more than compensates the intensive margin, as $I_H$ and $I_L$ experience a slight fall: given the unresponsiveness of $Q$ to the exogenous shock, the decrease in the vertical-innovation rates reflects essentially the shift of resources towards the extensive margin over transition. The independence of $Q$ relatively to the relative supply of skills implies that the relative number of firms, relative production and the relative firm size are
unchanged along the transition path, too.

**Industry dynamics: long-run effect** Eventually, both $g_{Q_L}$ and $g_{Q_H}$ will settle down at a level that is higher than the pre-shock steady state level, with the short-run effect of the exogenous shock translating almost one-to-one into the long-run effect. In the case of the relative number of firms, relative production and the relative firm size, the long-run effect results strictly from the short-run response to the exogenous shock.

**Aggregate dynamics** The growth rate and the real interest rate experience only a very slight increase along the transition path; thus, the long-run effect of an increase in $H$ results almost entirely from the short-run response to the exogenous shock, but which is smaller than in Scenario 1, since scale effects are partially removed from the model.

![Figure 4](image)

*Figure 4: Scenario 3: short-run effects and transitional dynamics of the aggregate and industry-level variables when $H$ increases from 0.3 to 0.5. $\epsilon = 0.5$."

### 3.2. Discussion and a simple calibration

It is noteworthy that, except for the knife-edge case in which market-size-channel and price-channel effects offset each other exactly (Scenario 3), as the economy evolves towards the new BGP, there is a noticeable shift of economic activity between sectors, specially in terms of production but also of the number of firms. For the baseline values
of the parameters considered in Section 3.1 and with $\epsilon = 0$ ($\epsilon = 1$), after a possible initial jump, relative production, $X$, and the relative number of firms, $N$, increase a total of, respectively, 15.2 and 10.3 percentage points (decrease 29.9 and 14.2 points) over 120 years. In contrast, the economic growth rate, $g$, and the real interest rate, $r$, remain roughly unchanged (after a discrete short-run adjustment, in Scenario 1). Since the economy takes a long time to reach the new BGP, we interpret these results for the aggregate variables as an approximation to the Kaldor facts along transitional dynamics, while structural change occurs due to nonbalanced sectoral growth. When the new BGP is reached, balanced growth at both the aggregate and the sectoral level is established, and thus no sector ever vanishes, in contrast to the result that may obtain in, e.g., Ngai and Pissarides (2007), Bonatti and Felice (2008), and Blankenau and Cassou (2009).

We would also like to emphasise that, as depicted by Figure 5, the speed of adjustment to a positive shock in the relative supply of skills may be quite different across variables, whether we compare them at the aggregate or the industry level. The speeds of convergence are also time-varying for each variable. The one with the slowest speed is clearly the relative number of firms, in contrast to relative production, a result that is mainly explained by the asymmetric impact of the complexity and congestion costs on vertical and horizontal R&D, which then implies different speeds of convergence of the two industry-structure variables towards the new BGP. On the other hand, when the market-size channel dominates, the interest rate converges at a higher speed than the economic growth rate, but both are slower than relative production. When the price-channel dominates, the fact that the interest rate follows a non-monotonic time-path that overshoots the new BGP (although only very slightly) after approximately 40 years implies that its speed of convergence will, at that time, become infinite; after passing through that point, the interest rate will eventually converge at a finite rate that is higher than that of the economic growth rate but smaller than that of relative production.

The importance of these features of transitional dynamics has been convincingly emphasised within the endogenous growth literature by Eicher and Turnovsky (2001). However, unlike the latter, we obtain flexible transitional dynamics without having to restrict our analysis to a non-scale version of our model, while the dimension of the dynamic system in detrended variables is the same in the two papers.

[Figure 5 goes about here]

---

29In the model, the shock in $H/L$ implies an immediate jump (the “short-run effect” analysed in Section 3.1) in some of the variables of interest. However, we are obviously conducting an artificial experiment by considering a one-off jump in $H/L$; in reality, the relative supply of skills should be expected to have followed a continuous time-path, even if at an accelerated rate, between the 80's and the 90's. Thus, more realistically, the short-run impact on those variables should be imagined as being spread out over a certain period of time, instead of as a discontinuous jump. Bearing this in mind, one may want to access the change in the variables of interest by considering both the discrete short-run adjustment and the ensuing transition path. In this case, and in particular under Scenario 1, the model predicts that the industry-structure ratios $X$ and $N$ raise a total of, respectively, 26.6 and 10.3 percentage points, while the aggregate variables $g$ and $r$ increase a total of, respectively, 0.5 and 0.78 percentage points, over 120 years.

30This accounts for the singularity observed in the lower panel of Figure 5 with respect to $\beta_r$. 

30
Finally, bearing in mind the available data at the sectoral level, something can be said about the adequacy of the theoretical results to the empirical side. According to the time series depicted by Figure 1, both measures of industry structure are growing over time, but with the former outpacing the latter. That is, the shift of economic activity occurs from the low- to the high-tech sectors and with a stronger impact on production than on the number of firms. This evidence suggests that Scenario 1 is the only one that is consistent with the empirical facts. As explained earlier, this scenario features the technological-knowledge bias working mainly through the market-size channel. A simple calibration exercise, summarised in Table 2, suggests that, under that scenario and under the hypothesis of an initial increase in relative supply of skills from 0.3 to 0.5, the model accounts for, respectively, 63 percent and 69 percent of the average annual growth rate of relative production and of the relative number of firms observed in the data from 1995 to 2007.

A prevailing market-size channel implies scale effects on industrial growth. This is in apparent contrast with the well-known endogenous-growth debate over the counterfactual character of scale effects. However, while the existing empirical results reject the existence of large scale effects in secular trend (e.g., Jones, 1995b), our quantitative results point out to the existence of scale effects in the medium term, in particular given the relatively short time span of the time-series data that we used in our calibration. Therefore, we
Relative production 
(annual growth rate) | Relative number of firms 
(annual growth rate) | SQE
---|---|---
\(\gamma = 1.2\) (baseline) | 0.63 | 0.08 | 0.0086
\(\gamma = 0.07\) (min SQE) | 0.77 | 0.48 | 0.0050
observed | 1.22 | 0.70 |

Table 2: Calibration exercise for the annual growth rate of relative production and of the relative firm size observed in the data (log differences of the weighed average across 14 European countries) over the period 1995-2007 (the longest period with available data for both variables; see further details in Figure 1). Predicted values for the growth rates are obtained by setting \(\epsilon = 0\), \(\zeta = 0.55\) and \(\sigma + \gamma = 2.4\). Parameter \(\gamma\) is chosen such that the model approximates the empirical annual growth rate of the industry structure variables by considering the numerically-obtained time paths for \(X\) and \(N\) from \(t = 5\) to \(t = 17\), where \(t = 0\) corresponds to the year of 1990. SQE = \(\sqrt{\sum_i(Z_{i \text{obs}} - Z_i)^2}\), \(i = 1, 2\), where \(Z_i\) \((Z_{i \text{obs}})\) denotes the predicted (observed) value of the annual growth rate of each of the industry structure variables.

believe that our results complement instead of antagonise the extant literature with this respect.

4. Concluding remarks

This paper builds an endogenous growth model of directed technical change with simultaneous vertical and horizontal R&D and flexible scale effects to study an analytical mechanism that is consistent with the coexistence of structural change and aggregate stability. In particular, we focus on shifts in the share of the high-vis-à-vis the low-tech sectors within manufacturing in the context of slow, but flexible, transitional dynamics. We show that, under the hypothesis of a positive shock in the proportion of high-skilled labour, the technological-knowledge bias channel leads to nonbalanced sectoral growth, while the aggregate variables are roughly unchanged. When the new BGP is (asymptotically) reached, balanced growth at both the aggregate and the sectoral level is established, and thus no sector ever vanishes, as seems to be the case empirically.

It is worth noting the asymmetric role played by the intensive and the extensive margin in explaining the time-path of the industry-level variables under scale and no-scale effects on growth. Our theoretical results show that a rich interaction between the two margins should be expected when one takes into account the short and medium-term responses to structural shocks. The fact that the shock in the relative supply of skills occur due to a rise in high-skilled labour paralleled by a stabilisation or only a slight decrease in low-skilled labour (which is in accordance to the empirical evidence) further enhances the asymmetry between the different scenarios for the degree of scale effects.
Under prevailing market-scale effects, the theoretical results are qualitatively consistent with the increase in the share of the high-tech sectors found in time-series data, computed as a weighted average across 14 European countries, whereas, at the aggregate level, the economic growth rate and the real interest rate remain approximately constant. We also presented a simple calibration exercise, which showed that the implied magnitudes for the shift in the share of the high-tech sectors are of around two-thirds of the change observed in the data from 1995 to 2007.

We leave for future research a full investigation of whether the analytical mechanism proposed in this paper plays a first-order role in nonbalanced sectoral growth at the empirical level, in particular by contrasting our mechanism with both the demand-side and the alternative supply-side mechanisms suggested by the extant literature.

References


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Appendix

A. Derivation of equation (33)

We derive the aggregate resource constraint from the households’ balance sheet and flow budget constraint. Firstly, consider the production function (1), such that, given threshold (6), we have firm n value product (time indexes are omitted)
\[ P(n) \cdot Y(n) = \begin{cases} 
  P(n) \cdot A \cdot \left[ \int_0^{N_L} \left( \lambda jL(\omega_l) \cdot X_L(n, \omega_L) \right)^{1-\alpha} d\omega_L \right] [(1-n) \cdot l \cdot L(n)]^\alpha, & 0 \leq n \leq \bar{n} \\
  P(n) \cdot A \cdot \left[ \int_0^{N_H} \left( \lambda jH(\omega_H) \cdot X_H(n, \omega_H) \right)^{1-\alpha} d\omega_H \right] [n \cdot h \cdot H(n)]^\alpha, & \bar{n} \leq n \leq 1 
\end{cases}\]  
(64)

Since, in equilibrium, the wage paid to each unit of human capital \( m \in \{L, H\} \) is equal to its marginal value product, we use (64) to get

\[ w_L = \frac{\partial (P(n)Y(n))}{\partial L(n)} \bigg|_{0 \leq n \leq \bar{n}} \Leftrightarrow w_L \cdot L(n) = \alpha \cdot P(n) \cdot Y(n) \bigg|_{0 \leq n \leq \bar{n}} \]  
\[ w_H = \frac{\partial (P(n)Y(n))}{\partial H(n)} \bigg|_{\bar{n} \leq n \leq 1} \Leftrightarrow w_H \cdot H(n) = \alpha \cdot P(n) \cdot Y(n) \bigg|_{\bar{n} \leq n \leq 1} \]  
(65)

Aggregating (65) across \( n \) and simplifying with the following labour-market clearing conditions

\[ \int_0^\bar{n} L(n)dn = L \Leftrightarrow L(n) = \frac{L}{\bar{n}} \]  
\[ \int_0^\bar{n} H(n)dn = H \Leftrightarrow H(n) = \frac{H}{1-\bar{n}} \]  
(66)

yields\(^{31}\)

\[ \int_0^\bar{n} w_L \cdot L(n)dn = \alpha \int_0^\bar{n} P(n) \cdot Y(n)dn \Leftrightarrow w_L L = \alpha Y_L \]  
\[ \int_0^1 w_H \cdot H(n)dn = \alpha \int_0^1 P(n) \cdot Y(n)dn \Leftrightarrow w_H H = \alpha Y_H \]  
(67)

where \( Y_L = \int_0^\bar{n} P(n)Y(n)dn \), \( Y_H = \int_0^1 P(n)Y(n)dn \), such that \( Y_{tot} \equiv Y_L + Y_H = \int_0^1 P(n)Y(n)dn \).

On the other hand, from the derivation of (10), we know that \( Y_L = \frac{1}{\alpha} \cdot (1-\alpha) \cdot P_L^\frac{2(1-\alpha)}{\alpha} \cdot Q_L \), \( Y_H = \frac{1}{\alpha} \cdot (1-\alpha) \cdot P_H^\frac{2(1-\alpha)}{\alpha} \cdot Q_H \). Also, from the derivation of (9), we learn that \( X_L = \int_0^{N_L} X_L(\omega_L)d\omega_L = \frac{1}{\alpha} \cdot (1-\alpha) \cdot P_L^{\frac{2}{\alpha}} \cdot Q_L \) and \( X_H = \int_0^{N_H} X_H(\omega_H)d\omega_H = \frac{1}{\alpha} \cdot (1-\alpha) \cdot P_H^{\frac{2}{\alpha}} \cdot Q_H \). Therefore, it is easy to show that, given (3),

\[ X_m = (1-\alpha)^2 Y_m \Leftrightarrow p_m X_m = (1-\alpha) Y_m, \ m \in \{L, H\} \]  
(68)

We put (67) and (68) together to get aggregate gross income

\[ Y_m = w_m m + pX_m, \ m \in \{L, H\} \]  
(69)

or, since total profits in each technology group are \( \pi_m N_m = \int_0^{N_m} (p_m(\omega_m) - 1) \cdot X_m(\omega_m)d\omega_m = (p - 1) \cdot X_m \), to get aggregate value added

\[ Y_m - X_m = w_m m + \pi_m N_m, \ m \in \{L, H\} \]  
(70)

\(^{31}\)Given Cobb-Douglas technology, expenditures across final goods are equalised (i.e., \( P(n)Y(n) \) is constant over \( n \)), which implies \( L(n) \) and \( H(n) \) constant over \( n \in [0, \bar{n}] \) and \( n \in [\bar{n}, 1] \), respectively.
Secondly, consider aggregate financial wealth held by all households in sector $m$, \( a_m = \int_0^{\infty} V_m(\omega_m)d\omega_m \). If we also consider the arbitrage condition between vertical and horizontal entry, we get
\[
a = a_L + a_H = \eta_L N_L + \eta_H N_H
\] (71)
which, by time-differentiation, becomes
\[
\dot{a} = \eta_L \dot{N}_L + \eta_H \dot{N}_H + \dot{\eta}_H N_H
\] (72)
Next, we solve (23), in the text, in order to $\dot{\eta}$ and, together with (71) and (30), substitute in (72) to get
\[
r(a_L + a_H) + w_LL + w_HH - C = \eta_L (r + I_L) N_L - \pi_L N_L + \eta_L \left( \frac{\dot{\pi}_L}{\pi_L} - \frac{1}{\alpha} \frac{\dot{P}_L}{P_L} \right) N_L + \\
+\eta_H (r + I_H) N_H - \pi_H N_H + \eta_H \left( \frac{\dot{\pi}_H}{\pi_H} - \frac{1}{\alpha} \frac{\dot{P}_H}{P_H} \right) N_H + \eta_L \dot{N}_L + \eta_H \dot{N}_H \Leftrightarrow \\
\Leftrightarrow w_LL + w_HH + \pi_L N_L + \pi_H N_H = C + I_L \eta_L N_L + I_H \eta_H N_H + \eta_L \dot{N}_L + \eta_H \dot{N}_H + \\
+ \left( \frac{\dot{\pi}_L}{\pi_L} - \frac{1}{\alpha} \frac{\dot{P}_L}{P_L} \right) \eta_L N_L + \left( \frac{\dot{\pi}_H}{\pi_H} - \frac{1}{\alpha} \frac{\dot{P}_H}{P_H} \right) \eta_H N_H
\] (73)
By using (70) in (73) and recalling $R_h = R_{hH} + R_{hL}$; $R_v = R_{vH} + R_{vL}$; $R_{hm} = \dot{\eta}_m N_m$ and $R_{vm} = I_m a_m + \left( \frac{\pi_m}{\pi_m} - \frac{1}{\alpha} \frac{\dot{P}_m}{\dot{P}_m} \right) \eta_m N_m$, we find
\[
Y_L - X_L + Y_H - X_H = C + R_{hH} + R_{hL} + R_{vH} + R_{vL} \Leftrightarrow \\
\Leftrightarrow Y_{tot} = X_{tot} + C + R_h + R_v
\]
which is (33).

Finally, observe that since the real interest rate $r$ consists of dividend payments in units of asset price minus the Poisson death rate, i.e., $r = \frac{\pi}{P} - I_m$, for each $t$ (see, e.g., (22) and (23)), then $a_m = V_m N_m \Rightarrow \pi_m N_m = (r + I_m)a_m$. From here and (70), we re-write (30) as
\[
\dot{a} = r(a_L + a_H) + w_LL + w_HH - C
\]
\[\text{To see this, recall that \( \frac{\pi_m}{\pi_m} - \frac{1}{\alpha} \frac{\dot{P}_m}{\dot{P}_m} = I_m \left( \frac{\alpha}{\lambda^{1+\alpha}} \right) \cdot ln\lambda \right) = I_m \left( \lambda^{1+\alpha} - 1 \right) \text{ for \( j_m = 1 \) and small \( \lambda \), and use it together with the consistency condition (24) solved in order to \( \eta_m N_m \), in (17), to get, again with \( m = L, R_{LH} = I_L \xi_L \lambda^{1-\alpha} Q_L = I_L a_L + \left( \frac{\dot{\pi}_L}{\pi_L} - \frac{1}{\alpha} \frac{\dot{P}_L}{P_L} \right) a_L, \right) \}
\]
\[ (\pi_L - I_L) a_L + (\pi_H - I_H) a_H + w_L L + w_H H - C = \]
\[ = Y_L - X_L + Y_H - X_H - I_L a_L - I_H a_H - C \]  \hspace{1cm} (74)

If we replace (33), solved in order to \( R_v + R_h \), in (74), we obtain
\[ \dot{a} = R_v + R_h - (I_L a_L + I_H a_H) \]  \hspace{1cm} (75)

which is the accumulation equation for \( a \). The first two terms on the right-hand side of (75) represent the *gross investment* in technological knowledge at time \( t \), whereas the third term represents the *depreciation* (obsolescence) of the existing technological knowledge stock due to the stochastic arrival of vertical innovations (i.e., as \( j \) jumps to \( j + 1 \)) in each technological group.

**B. Two-dimensional phase diagrams in \((x_L, x_H)\) space**

The phase diagrams in Figure 6 are two-dimensional projections in \((x_L, x_H)\) space, plotted under three scenarios for \( \epsilon \) (\( \epsilon = 0, \epsilon = 1, \epsilon = 0.5 \)). The lower panels depict the trajectories of \( x_L \) and \( x_H \) from \( t = 1 \) to \( t = 120 \) (the same number of periods as in Figures 2-5), while the upper panels show the switching curves \( I_L(x_L, x_H) = 0 \) and \( I_H(x_L, x_H) = 0 \). Since these locus move as \( Q(t) \) and \( z(t) \) converge towards the new BGP, we considered \( Q(t) \) and \( z(t) \) valued at \( t = 1 \) and \( t = 120 \) to make the planar representation of the switching curves tractable. As a consequence, two pairs of switching curves appear in each scenario. Those curves divide the state space into three zones: in the northeast area, where \( I_m(x_L, x_H) < 0 \), \( m \in \{L, H\} \), the dynamics will be given by the dynamic system (41)-(44) by setting \( I_m^L = 0 \) according to equation (36); in the southwest area, where \( I_m(x_L, x_H) > 0 \), \( m \in \{L, H\} \), the dynamics is given by setting \( I_m^H = I_m \); in the area between the two switching curves (which exists if \( \epsilon \neq 0.5 \)), we either have \( I_H(x_L, x_H) > 0 \) and \( I_L(x_L, x_H) < 0 \), and thus \( I_H^L = I_H \) and \( I_L^H = 0 \), or \( I_H(x_L, x_H) < 0 \) and \( I_L(x_L, x_H) > 0 \), and thus \( I_H^L = 0 \) and \( I_L^H = I_L \). Figure 6 shows that, for our numerical simulations and given the considered shock in the relative supply of skills, the saddle-path trajectories for \( x_L \) and \( x_H \) never cross the locus \( I_L(x_L, x_H) = 0 \) and \( I_H(x_L, x_H) = 0 \) and thus never leave the southwest area of the phase diagram.
Figure 6: Two-dimensional transition paths in \((x_L, x_H)\) space under three scenarios for \(\epsilon\) (respectively, \(\epsilon = 0, \epsilon = 1, \epsilon = 0.5\)). The two panels in each column depict two parts of the same phase diagram; we resorted to separate panels to accommodate the very different scale along the vertical axis that corresponds to the \(I_m(x_L, x_H) = 0, \ m \in \{L, H\}\), loci and to the trajectories of \(x_L\) and \(x_H\).
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<td>477</td>
<td>Theory of Collusion in the Labor Market</td>
<td>Pedro Gonzaga, António Brandão and Hélder Vasconcelos</td>
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<td>476</td>
<td>Political Economy and the 'Modern View' as reflected in the History of Economic Thought</td>
<td>Mário Graça Moura and António Almodovar</td>
<td>December 2012</td>
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