

# **On rank estimation in symmetric matrices: the case of indefinite matrix estimators**

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# On rank estimation in symmetric matrices: the case of indefinite matrix estimators

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## Abstract

We focus on the problem of rank estimation in an unknown symmetric matrix based on a symmetric, asymptotically normal estimator of the matrix. The related positive definite limit covariance matrix is assumed to be estimated consistently, and to have either a Kronecker product or an arbitrary structure. These assumptions are standard although they also exclude the case when the matrix estimator is positive or negative semidefinite. We adapt and reexamine here some available rank tests, and introduce a new rank test based on the eigenvalues of the matrix estimator. We discuss several applications where rank estimation in symmetric matrices is of interest, and also provide a small simulation study and an application.

**Keywords:** rank, symmetric matrix, eigenvalues, matrix decompositions, estimation, asymptotic normality, consistency.

**JEL classification:** C12, C13.

## 1 Introduction

Let  $M$  be an unknown  $n \times k$  matrix,  $\widehat{M}$  be its estimator and  $\text{rk}\{M\}$  denote the rank of the matrix  $M$ . Suppose that  $\widehat{M}$  is asymptotically normal having the limiting covariance matrix  $C$  which can be consistently estimated by  $\widehat{C}$ . Under these assumptions, many authors have proposed ways to estimate the rank of the unknown matrix  $M$  by using the estimators  $\widehat{M}$  and  $\widehat{C}$ . The recent and commonly used rank tests are the LDU (Lower-Diagonal-Upper triangular decomposition) test of Gill and Lewbel (1992), Cragg and Donald (1996), the Minimum Chi-Squared (MINCHI2) test of Cragg and Donald (1997), the ALS (Asymptotic Least Squares) test of Gouriéroux, Monfort and Trognon (1985) and Robin and Smith (1995), the SVD (Singular Value Decomposition) tests in Ratsimalahelo (2002, 2003) and Kleibergen and Paap (2003), or the characteristic root test of Robin and Smith (1995, 2000). The aforementioned papers also discuss many situations where the rank estimation is of interest, for example, in the context of factor and state-space models, cointegration and the theory of demand systems. See also Anderson (1951), Camba-Mendez, Kapetanios, Smith and Weale (2003),

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Camba-Mendez and Kapetanios (2001), Donkers and Schafgans (2003), Lewbel (1991), Donald (1997), Fortuna (2004), Bura and Cook (2001, 2003) and others. The limiting covariance matrix  $C$  of  $\widehat{M}$  can have either a Kronecker product structure  $C = A \otimes B$  or a general, non-Kronecker product structure. The Kronecker product structure of  $C$  allows, for example, to obtain simpler expressions of the related rank test statistics and to establish their exact asymptotics.

In some applications (see Section 3 below) the unknown matrix  $M$  and its estimator  $\widehat{M}$  are *symmetric*. It is generally believed that, after a few slight modifications, available rank tests work for symmetric matrices as well. For example, the MINCHI2 statistic for the rank test of  $H_0 : \text{rk}\{M\} = r$  with  $n \times n$  symmetric matrices  $M, \widehat{M}$  follows a  $\chi^2$  distribution with  $(n - r)(n - r + 1)/2$  degrees of freedom, whereas the limiting  $\chi^2$  distribution has  $(n - r)^2$  degrees of freedom in the case of arbitrary, nonsymmetric square matrices  $M, \widehat{M}$  (Cragg and Donald (1997), p. 235, Robin and Smith (1995)). Though rank tests work for symmetric matrices as well, the related results are scattered through the literature and are often discussed superficially. One of our goals is to gather various available rank tests for symmetric matrices. Another goal is to shed additional light on some of these tests.

We suppose below that there is a symmetric, asymptotically normal estimator  $\widehat{M}$  of the unknown matrix  $M$ . In fact, we use two different sets of assumptions, Assumptions (A) and (A\*) in Section 2 below, expressing the asymptotic normality condition. Assumptions (A) involve a limit covariance-like matrix which has a Kronecker product structure. These assumptions arise in some but not all applications involving symmetric matrices. Assumptions (A\*) are more general as they allow the limit covariance matrix associated with  $\widehat{M}$  to be arbitrary. These assumptions are those commonly encountered in the Statistics and Econometrics literature. We should point out, however, that they also exclude the cases when the matrix  $\widehat{M}$  is, in addition, positive or negative semidefinite (see Proposition 2.3 and the discussion following it below).

Given either Assumptions (A) or (A\*), we shall discuss below the MINCHI2, LDU and SVD rank tests for symmetric matrices. In particular, under stronger Assumptions (A), we shall establish an exact asymptotics of the related MINCHI2 statistic. In the case of the LDU test for symmetric matrices, we shall introduce the symmetric pivoting procedure to replace the complete pivoting used in the Gaussian elimination of unrestricted matrices.

In addition to well-known rank tests above, we shall introduce a rank test based on the eigenvalues of the symmetric matrices themselves. Moreover, we will do so only for the case of symmetric, positive or negative semidefinite matrices  $M$ . Observe also that symmetric matrices are special among all matrices: it is possible to define eigenvalues only for square matrices and the eigenvalues of a square matrix are always real when the matrix is symmetric. The resulting EIG rank test is easy to formulate under stronger Assumptions (A), and becomes more involved when Assumptions (A\*) are used. Under Assumptions (A), the EIG rank test was already used, albeit implicitly, in Donald (1997) and Fortuna (2004). We state it here in general and also prove it under more general assumptions.

We shall focus throughout the paper on rank tests, that is, tests for  $H_0 : \text{rk}\{M\} \leq r$  against  $H_1 : \text{rk}\{M\} > r$ , rather than on estimation of  $\text{rk}\{M\}$  itself. The latter follows from the rank tests in a standard way, for example, by using sequential testing (see Section 3 in Cragg and Donald (1997), Section 5 in Robin and Smith (2000)), or the model selection approach. Observe also that, because of the one-sided nature of the rank tests, one often writes  $H_0 : \text{rk}\{M\} = r$  instead of  $H_0 : \text{rk}\{M\} \leq r$ .

The paper is organized as follows. In Section 2, we state and discuss the assumptions used in this work. Some situations where rank estimation in symmetric matrices arise, are described in Section 3. The EIG rank test is given in Section 4. In particular, this test sheds light on the MINCHI2 test of Section 5. The LDU and SVD rank tests are considered in Sections 6 and 7, respectively. A simulation study and an application can be found in Sections 8 and 9, respectively. The proofs of all technical

results are contained in Appendix A.

## 2 Assumptions and other preliminaries

Suppose that  $M$  is an unknown *symmetric*  $n \times n$  matrix with real entries. We are interested in estimating the rank of the matrix  $M$  by using its estimator  $\widehat{M} = \widehat{M}(N)$  where  $N$  is the sample size or any other relevant parameter such that  $N \rightarrow \infty$ . Let  $\mathcal{Z}_n$  be a symmetric  $n \times n$  matrix having independent normal entries with variance 1 on the diagonal and variance 1/2 off the diagonal. Our first assumption concerns the limiting distribution of an appropriately normalized estimator of the unknown matrix.

**Assumptions (A).** There are real, symmetric  $n \times n$  matrices  $\widehat{M} = \widehat{M}(N)$  such that

$$\sqrt{N}F(\widehat{M} - M)F' \xrightarrow{d} \mathcal{Z}_n \quad (2.1)$$

as  $N \rightarrow \infty$ , where  $\xrightarrow{d}$  stands for the convergence in distribution and  $F$  is a nonsingular  $n \times n$  matrix. There are matrices  $\widehat{F}$  such that

$$\widehat{F} \xrightarrow{p} F, \quad (2.2)$$

where  $\xrightarrow{p}$  stands for the convergence in probability.

Letting  $\text{vec}$  and  $\otimes$  denote the standard vec operator and the Kronecker product, respectively, the condition (2.1) can be expressed as  $\sqrt{N}(F \otimes F)\text{vec}(\widehat{M} - M) \xrightarrow{d} \text{vec}(\mathcal{Z}_n)$  or

$$\sqrt{N}\text{vec}(\widehat{M} - M) \xrightarrow{d} (F^{-1} \otimes F^{-1})\text{vec}(\mathcal{Z}_n). \quad (2.3)$$

Observe that the matrix  $F^{-1} \otimes F^{-1}$  on the right-hand side of (2.3) has a Kronecker product structure. Observe also that, since the matrices  $\widehat{M}$  and  $M$  are symmetric, the vector  $\sqrt{N}\text{vec}(\widehat{M} - M)$  has a singular limiting covariance matrix. One can avoid singularity by using the standard vech operator which stacks in a column only the lower triangular part of a symmetric matrix.

Let also  $D_n$  and  $K_n$  be the  $n^2 \times n(n+1)/2$  *duplication matrix* and the  $n^2 \times n^2$  *commutation matrix*, respectively, satisfying

$$\text{vec}(A) = D_n \text{vech}(A), \quad K_n \text{vec}(B) = \text{vec}(B'), \quad (2.4)$$

for any  $n \times n$  symmetric matrix  $A$  and any  $n \times n$  matrix  $B$  (see, for example, Magnus and Neudecker (1999), pp. 46–53). It is known that  $D'_n D_n$  is nonsingular and that

$$D_n^+ = (D'_n D_n)^{-1} D'_n \quad (2.5)$$

is the Moore-Penrose inverse of the matrix  $D_n$ . The matrix  $D_n^+$  is  $n(n+1)/2 \times n^2$ , satisfies  $\text{vech}(A) = D_n^+ \text{vec}(A)$  for a symmetric matrix  $A$  and is also known as the *elimination matrix*. The matrices  $D_n$ ,  $D_n^+$  and  $K_n$  arise naturally in the context of symmetric matrices and will be used extensively in the presentation below.

We can now express the condition (2.3) in terms of the vech operator and a nonsingular limit covariance matrix. Observe that

$$\mathcal{Z}_n \stackrel{d}{=} \frac{\mathcal{X}_n + \mathcal{X}'_n}{2} \quad (2.6)$$

with  $\mathcal{X}_n \stackrel{d}{=} \mathcal{N}(0, I_{n^2})$  and hence, by using (2.4) and Theorem 12,(b), in Magnus and Neudecker (1999), p. 49,

$$\text{vec}(\mathcal{Z}_n) \stackrel{d}{=} \mathcal{N}(0, \Omega) \quad \text{with} \quad \Omega = \frac{1}{2}(I_{n^2} + K_n) = D_n D_n^+. \quad (2.7)$$

Then, the vector  $(F^{-1} \otimes F^{-1})\text{vec}(\mathcal{Z}_n)$  in (2.3) has the covariance matrix  $(F^{-1} \otimes F^{-1})D_n D_n^+ (F^{-1} \otimes F^{-1})$ . Writing  $\text{vec}(\widehat{M} - M) = D_n \text{vech}(\widehat{M} - M)$  in (2.3) and pre-multiplying (2.3) by  $D_n^+$  so that  $D_n^+ D_n = I_{n^2}$ , we obtain that the limit covariance matrix of  $\sqrt{N} \text{vech}(\widehat{M} - M)$  is

$$D_n^+ (F^{-1} \otimes F^{-1}) D_n D_n^+ (F^{-1} \otimes F^{-1}) D_n^{+'}. \quad (2.8)$$

We may now use Theorem 13,(b) and (d), in Magnus and Neudecker (1999), pp. 49-50, to reexpress (2.8) as in the following proposition.

**Proposition 2.1** *The condition (2.1) or (2.2) is equivalent to*

$$\sqrt{N} \text{vech}(\widehat{M} - M) \xrightarrow{d} \mathcal{N}(0, C),$$

where

$$C = D_n^+ ((F'F)^{-1} \otimes (F'F)^{-1}) D_n^{+'} = (D_n' (F'F \otimes F'F) D_n)^{-1}. \quad (2.9)$$

Observe that the covariance  $C$  in (2.9) has a Kronecker product like structure. More generally, as commonly found in the Statistics and Econometrics literature, one can assume that the matrix  $C$  is arbitrary.

**Assumptions (A\*).** There are symmetric matrices  $\widehat{M}$  such that

$$\sqrt{N} \text{vech}(\widehat{M} - M) \xrightarrow{d} \mathcal{N}(0, C), \quad (2.10)$$

where the matrix  $C$  is positive definite. There are also matrices  $\widehat{C}$  such that

$$\widehat{C} \xrightarrow{p} C. \quad (2.11)$$

In the sequel, we shall use either Assumptions (A) or (A\*), and refer to Assumptions (A) by saying that the related limit covariance matrix has a Kronecker product structure. We shall also use (2.10) expressed as

$$\sqrt{N}(\widehat{M} - M) \xrightarrow{d} \mathcal{Y}, \quad (2.12)$$

where  $\mathcal{Y}$  is such that

$$\text{vec}(\mathcal{Y}) \stackrel{d}{=} \mathcal{N}(0, W) \quad \text{with} \quad W = D_n C D_n'. \quad (2.13)$$

The matrix  $W$  is singular because  $\widehat{M}$  and  $M$  are symmetric.

Observe also that

$$\sqrt{N} F(\widehat{M} - M) G' \xrightarrow{d} \mathcal{Z}_n, \quad (2.14)$$

where  $G$  is another nonsingular matrix, may seem a more general Kronecker product structure than that used in (2.1). The next proposition shows that the condition (2.14) can be reduced to (2.1). This proposition is proved in Appendix A. Its proofs relies solely on the symmetry of  $\widehat{M}$  and  $M$ .

**Proposition 2.2** *If (2.14) holds, then  $G = cF$  for some  $c \in \mathbb{R} \setminus \{0\}$ . Hence, the condition (2.14) can be reduced to (2.1) with  $F$  replaced by  $\sqrt{|c|}F$ .*

Let us also point out another important consequence of Assumptions (A) and (A\*). Under these assumptions, the estimator  $\widehat{M}$  cannot be positive or negative semidefinite.

**Proposition 2.3** *If Assumptions (A\*) hold with a positive definite matrix  $C$  and  $\text{rk}\{M\} < n$ , then  $\widehat{M}$  cannot be positive or negative semidefinite.*

The proposition shows that Assumptions (A\*) exclude some interesting cases of rank estimation in symmetric matrices, for example, the case when  $M = \Sigma$  and  $\widehat{M} = \widehat{\Sigma}$  are theoretical and sample covariance matrices. These problems should therefore be addressed through a different approach. Some authors have, in fact, incorrectly assumed that it was possible to have rank deficiency for a semidefinite matrix and at the same time have a positive definite variance covariance matrix, see Cragg and Donald (1997) and Donkers and Schafgans (2003).

Proposition 2.3 does not exclude the case where the unknown matrix  $M$  is positive semidefinite. Example 3.1 below shows that this may indeed occur. Our EIG rank test is formulated in Section 4, in fact, only under this assumption.

**Assumption (B).** The unknown symmetric matrix  $M$  is positive semidefinite.

### 3 Examples

We discuss here two examples involving estimation the rank of an unknown symmetric matrix under Assumptions (A) or (A\*).

**Example 3.1** (Number of factors in a nonparametric relationship.) Consider a multivariate nonparametric relationship

$$Y_i = F(X_i) + \epsilon_i, \quad i = 1, \dots, N, \quad (3.1)$$

between a  $n \times 1$  vector  $Y_i$  of response variables and a  $d \times 1$  vector  $X_i$  of independent variables, where  $F$  is a vector of unknown functions and  $\epsilon_i$  are perturbation terms. Suppose that  $E(\epsilon_i|X_i) = 0$  and  $E(\epsilon_i\epsilon_i'|X_i) = \Sigma$  with a nonsingular matrix  $\Sigma$ . Let  $x^1$  be a  $d_1 \times 1$  (possibly empty) subvector of a  $d \times 1$  vector  $x$ . Define  $\text{rk}\{F\} = \text{rk}\{F; x^1\}$  as the smallest integer  $r$  for which the function  $F(x)$  can be expressed as

$$F(x) = \Theta'x^1 + A \cdot H(x), \quad (3.2)$$

where  $\Theta'$  is a  $n \times d_1$  matrix,  $A$  is a  $n \times r$  matrix and  $H(x)$  is a  $r \times 1$  vector of functions of  $x$ . Many econometric problems, for example, related to demand systems, nonparametric instrumental variables and arbitrage pricing, can be formulated as inference of  $\text{rk}\{F\}$  for a suitable choice of  $x^1$  (Donald (1997)).

Estimation of  $\text{rk}\{F\}$  is, in fact, equivalent to estimation of rank of a symmetric matrix under Assumptions (A). For simplicity, we shall focus only on the kernel based tests which are one type of tests considered by Donald (1997). Arguing as in the proof of Lemma 2.1 of Fortuna (2004), one can show that

$$\text{rk}\{F\} = \text{rk}\{\Gamma_w\}, \quad (3.3)$$

where  $\Gamma_w$  is a  $n \times n$  matrix defined by

$$\Gamma_w = Ep(X_i)\tilde{F}(X_i)\tilde{F}(X_i)' \quad (3.4)$$

with a density function  $p(x)$  of  $X_i$  and  $\tilde{F}(x) = F(x) - EF(X_i)X_i^1'E(X_i^1X_i^{1'})^{-1}x^1$ . Observe that the matrix  $\Gamma_w$  is *symmetric* and *positive semidefinite*. Following Donald (1997), it is natural to estimate the matrix  $\Gamma_w$  by

$$\widehat{\Gamma}_w = \frac{1}{N(N-1)} \sum_{i \neq j} (Y_i - \widehat{\Pi}'_1 X_i^1)(Y_j - \widehat{\Pi}'_1 X_j^1)' K_h(X_i - X_j), \quad (3.5)$$

where  $\widehat{\Pi}_1 = (X^{1'}X^1)^{-1}X^{1'}Y$  and  $K_h(x) = h^{-d}K(x/h)$  denotes a kernel  $K$  scaled by a bandwidth  $h > 0$ . Observe that the estimator  $\widehat{M}$  is symmetric as well but also indefinite. Following the proof of Lemma 2 in Donald (1997), one can show that, under suitable assumptions on  $X_i$ ,  $\epsilon_i$  and  $F$ ,

$$VNh^{d/2}\Sigma^{-1/2}(\widehat{\Gamma}_w - \Gamma_w)\Sigma^{-1/2} \xrightarrow{d} \mathcal{Z}_n, \quad (3.6)$$

where  $V = (2\|K\|_2^2 Ep(X_i))^{-1/2}$  and  $\|K\|_2^2 = \int K(x)^2 dx$ , and that the covariance matrix  $\Sigma$  can be estimated consistently.

Relations (3.3)–(3.6) show that estimation of  $\text{rk}\{F\}$  is equivalent to rank estimation of an unknown symmetric matrix  $\Gamma_w$  under Assumptions (A). In fact, Theorem 2 in Donald (1997) is a special case of the more general EIG rank test considered in Section 4 below.

When  $x^1$  is empty, estimation of  $\text{rk}\{M\}$  is also considered by Kneip (1994). An extension of Donald (1997) to estimation of the so-called local rank in a nonparametric relationship can be found in Fortuna (2004).

Estimation of the number of factors is considered above under Assumptions (A). More general Assumptions (A\*) arise naturally with heteroskedastic errors in the following sense. Assume the model (3.1) but suppose that  $E(\epsilon_i \epsilon_i' | X_i) = \Sigma(X_i)$  with nonsingular matrices  $\Sigma(x)$ . Under this and additional assumptions on  $X_i$ ,  $\epsilon_i$ ,  $F$  and  $\Sigma$ , we expect that

$$Nh^{d/2}\text{vech}(\widehat{\Gamma}_w - \Gamma_w) \xrightarrow{d} \mathcal{N}(0, C), \quad (3.7)$$

where

$$C = 2\|K\|_2^2 D_n^+ E\left((\Sigma(X_i) \otimes \Sigma(X_i))p(X_i)\right) D_n D_n^+ =: 2\|K\|_2^2 D_n^+ B D_n D_n^+ D_n^{+'}. \quad (3.8)$$

These relations are derived based on computation of the limiting covariance matrix of  $Nh^{d/2}\text{vech}(\widehat{\Gamma}_w - \Gamma_w)$ . When  $\Sigma(x) \equiv \Sigma$ , observe that (3.7) becomes (3.6). In practice, we expect that the matrix  $B$  in (3.8) can be consistently estimated through

$$\widehat{B} = \frac{1}{N(N-1)} \sum_{i \neq j} \left( (Y_i - \widehat{F}(X_i))(Y_i - \widehat{F}(X_i))' \right) \otimes \left( (Y_j - \widehat{F}(X_j))(Y_j - \widehat{F}(X_j))' \right) K_h(X_i - X_j), \quad (3.9)$$

where  $\widehat{F}(x)$  is a standard kernel-based estimator of  $F(x)$ . The idea behind (3.9) is that  $\widehat{B} \approx E(\epsilon_i \epsilon_i' \otimes \epsilon_j \epsilon_j') K_h(X_i - X_j) \approx B, i \neq j$ .

**Example 3.2** (Reduced rank regression with two sets of regressors.) Consider a multivariate regression model with two sets of regressors

$$Y_k = AX_k + BZ_k + \epsilon_k, \quad k = 1, \dots, N, \quad (3.10)$$

where  $Y_k$  is a  $n \times 1$  vector of response variables,  $X_k$  and  $Z_k$  are two sets of  $p \times 1$  and  $q \times 1$  regressors, and  $A$  and  $B$  are unknown  $n \times p$  and  $n \times q$  matrices. Suppose that  $E(\epsilon_i) = 0$  and  $E(\epsilon_i \epsilon_i') = \Sigma$  with a nonsingular matrix  $\Sigma$ . In some applications,  $B$  is restricted to be symmetric, and the goal is to estimate its rank. An example can be found in the context of demand systems, where  $Y_k$  is a vector of budget shares of a household  $k$ ,  $X_k$  is the vector of the logarithm of real total expenditures and other relevant variables, and  $Z_k$  is the vector of relative prices (Robin and Smith (2000), p. 161).

Imposing the symmetry restriction, the matrix  $B$  can be estimated by a symmetric matrix  $\widehat{B}$  obtained through the usual least squares. Under standard assumptions on  $X_i$ ,  $Z_i$  and  $\epsilon_i$ , one can show that

$$\sqrt{N}\text{vech}(\widehat{B} - B) \xrightarrow{d} \mathcal{N}(0, C), \quad (3.11)$$

where  $C$  is nonsingular. The exact expressions for  $\widehat{B}$  and  $C$  are quite lengthy. In a special case when  $A$  is empty, for example, we have

$$\text{vech}(\widehat{B}) = \left( D'_n(ZZ' \otimes I_n)D_n \right)^{-1} D'_n \text{vec}(YZ') \quad (3.12)$$

and, under suitable assumptions on  $Z_i, \epsilon_i$ ,

$$C = \left( D'_n(EZ_iZ'_i \otimes I_n)D_n \right)^{-1} \left( D'_n(EZ_iZ'_i \otimes \Sigma)D_n \right) \left( D'_n(EZ_iZ'_i \otimes I_n)D_n \right)^{-1}, \quad (3.13)$$

where  $Z = (Z_1 \dots Z_N)$  and  $Y = (Y_1 \dots Y_N)$ . The asymptotic relation (3.11) shows that  $\text{rk}\{B\}$  can be estimated under Assumptions  $(A^*)$  by one of the rank tests considered in this paper.

## 4 EIG rank test for semidefinite matrices

We shall provide here a rank test for a symmetric, positive semidefinite matrix  $M$  based on eigenvalues. Suppose first that Assumptions  $(A)$  of Section 2 hold. Let

$$\widehat{\lambda}_1 \leq \dots \leq \widehat{\lambda}_n \quad (4.1)$$

be the ordered eigenvalues of the matrix  $\widehat{M}\widehat{F}'\widehat{F}$ . Observe that these eigenvalues are real because  $\widehat{M}$  is symmetric and  $\widehat{F}'\widehat{F}$  is (asymptotically) positive definite. If  $\text{rk}\{M\} = \text{rk}\{MF'F\} = q$  and, without loss of generality,  $M$  is positive semidefinite, then the ordered eigenvalues of  $MF'F$  are  $0 = \lambda_1 = \dots = \lambda_{n-q} < \lambda_{n-q+1} \leq \dots \leq \lambda_n$ . Since  $\widehat{M}\widehat{F}'\widehat{F} \rightarrow_p MF'F$ , we have  $\widehat{\lambda}_k \rightarrow_p \lambda_k$ ,  $k = 1, \dots, n$ , and hence  $\widehat{\lambda}_k \rightarrow_p 0$ ,  $k = 1, \dots, n - q$ . The following theorem provides the exact asymptotics of  $\widehat{\lambda}_k$ ,  $k = 1, \dots, n - q$ . Let

$$\gamma_1(\mathcal{Z}_k) \leq \dots \leq \gamma_k(\mathcal{Z}_k)$$

be the ordered eigenvalues of the matrix  $\mathcal{Z}_k$  defined before Assumptions  $(A)$ .

**Theorem 4.1** *Suppose that Assumptions  $(A)$  and  $(B)$  hold. Let  $\text{rk}\{M\} = q$ , and  $\widehat{\lambda}_k$ ,  $k = 1, \dots, n$ , be the ordered eigenvalues defined in (4.1). Then,*

$$\left( \sqrt{N}\widehat{\lambda}_k \right)_{k=1}^{n-q} \xrightarrow{d} \left( \gamma_k(\mathcal{Z}_{n-q}) \right)_{k=1}^{n-q}, \quad (4.2)$$

$$\sqrt{N}\widehat{\lambda}_k \xrightarrow{p} +\infty, \quad k = n - q + 1, \dots, n. \quad (4.3)$$

Theorem 4.1 is proved in Appendix A. It is seen from the proof of the theorem that  $\widehat{F}'\widehat{F}$  in  $\widehat{M}\widehat{F}'\widehat{F}$  allows to obtain standardized limit laws for the eigenvalues.

Theorem 4.1 can be used to construct a test for the rank of the matrix  $M$  in the following way. Consider the test statistic

$$\widehat{\xi}_{\text{eig}}(r) = \sqrt{\frac{N}{n-r}} \sum_{k=1}^{n-r} \widehat{\lambda}_k, \quad r = 0, \dots, n. \quad (4.4)$$

The following theorem, proved in Appendix A, shows that the test statistic (4.4) can be used to test for  $H_0 : \text{rk}\{M\} \leq r$  in a standard way. The statistic (4.4) was used by Donald (1997) and Fortuna (2004) to infer the number of factors in a nonparametric relationship (see Section 3). The stochastic dominance  $\xi \leq_d \eta$  below stands for  $P(\xi > x) \leq P(\eta > x)$  for any  $x \in \mathbb{R}$ .

**Theorem 4.2** Suppose that Assumptions (A) and (B) hold, and let  $\widehat{\xi}_{\text{eig}}(r)$  be the test statistic defined by (4.4). Then, under  $H_0 : \text{rk}\{M\} \leq r$ , we have

$$\widehat{\xi}_{\text{eig}}(r) \xrightarrow{d} \frac{1}{\sqrt{n-r}} \sum_{k=1}^{n-r} \gamma_k(\mathcal{Z}_{n-q}) \stackrel{d}{\leq} \mathcal{N}(0, 1), \quad (4.5)$$

where  $q = \text{rk}\{M\}$  and  $\leq_d$  becomes  $=_d$  when  $r = q$ . Under  $H_1 : \text{rk}\{M\} > r$ , we have  $\widehat{\xi}_{\text{eig}}(r) \rightarrow_p +\infty$ .

**Remark 4.1** Assumption (B) can be dropped from Theorem 4.1. If the eigenvalues of  $M$  are

$$\lambda_1 \leq \dots \leq \lambda_{l-1} < 0 = \lambda_l = \dots = \lambda_{l+n-q-1} < \lambda_{l+n-q} \leq \dots \leq \lambda_n \quad (4.6)$$

with  $q = \text{rk}\{M\}$ , then one can show as in the proof of Theorem 4.1 that

$$\sqrt{N}\widehat{\lambda}_k \xrightarrow{p} -\infty, \quad k = 1, \dots, l-1, \quad (4.7)$$

$$\left(\sqrt{N}\widehat{\lambda}_k\right)_{k=l}^{l+n-q-1} \xrightarrow{d} \left(\gamma_k(\mathcal{Z}_{n-q})\right)_{k=1}^{n-q} \quad (4.8)$$

and

$$\sqrt{N}\widehat{\lambda}_k \xrightarrow{p} +\infty, \quad k = l+n-q, \dots, n. \quad (4.9)$$

Without imposing Assumption (B), however, we believe there is no simple way to define a test statistic similar to (4.4). In that case we suggest that one should use one of the other rank tests.

We were able to obtain standardized limit laws in Theorems 4.1 and 4.2 after properly normalizing the eigenvalues of  $\widehat{M}$ . It is not clear that such a normalization is available under more general Assumptions ( $A^*$ ). We can nevertheless establish the asymptotics of the eigenvalues of  $\widehat{M}$  and then use this to formulate the corresponding rank test.

Let

$$\widehat{v}_1 \leq \dots \leq \widehat{v}_n \quad (4.10)$$

be the ordered eigenvalues of the matrix  $\widehat{M}$ , and  $0 = v_1 = \dots = v_q < v_{q+1} \leq \dots \leq v_n$  be the ordered eigenvalues of the matrix  $M$  with  $q = \text{rk}\{M\}$ . Let  $U$  be an  $n \times n$  orthogonal matrix in the Schur decomposition of the matrix  $M$ :

$$\text{diag}(v_n, \dots, v_1) = U' M U = \begin{pmatrix} U_1' \\ U_2' \end{pmatrix} M \begin{pmatrix} U_1 & U_2 \end{pmatrix}, \quad (4.11)$$

where  $U$  is partitioned into  $n \times r$  matrix  $U_1$  and  $n \times (n-r)$  matrix  $U_2$ . Also let

$$\gamma_1(U_2' \mathcal{Y} U_2) \leq \dots \leq \gamma_{n-r}(U_2' \mathcal{Y} U_2)$$

be the ordered eigenvalues of the matrix  $U_2' \mathcal{Y} U_2$ , where  $\mathcal{Y}$  is the matrix defined in (2.12).

**Theorem 4.3** Suppose that Assumptions ( $A^*$ ) and (B) hold. Let  $\text{rk}\{M\} = q$ ,  $U_2$  be defined by (4.11) with  $r = q$ , and  $\widehat{v}_k$ ,  $k = 1, \dots, n$ , be the ordered eigenvalues defined in (4.10). Then,

$$\left(\sqrt{N}\widehat{v}_k\right)_{k=1}^{n-q} \xrightarrow{d} \left(\gamma_k(U_2' \mathcal{Y} U_2)\right)_{k=1}^{n-q}, \quad (4.12)$$

$$\sqrt{N}\widehat{v}_k \xrightarrow{p} +\infty, \quad k = n-q+1, \dots, n. \quad (4.13)$$

Consider the test statistic

$$\widehat{\eta}_{\text{eig}}(r) = \sqrt{N} \sum_{k=1}^{n-r} \widehat{v}_k, \quad r = 0, \dots, n. \quad (4.14)$$

By using Theorem 4.3 and the relation (2.13), we see that, under  $H_0 : \text{rk}\{M\} = r$ ,

$$\widehat{\eta}_{\text{eig}}(r) \xrightarrow{d} \sum_{k=1}^{n-r} \gamma_k(U_2' \mathcal{Y} U_2) = \text{tr}\{U_2' \mathcal{Y} U_2\} \stackrel{d}{=} \mathcal{N}(0, \sigma_r^2), \quad (4.15)$$

where  $\sigma_r^2 = \text{vec}(U_2 U_2')' D_n C D_n' \text{vec}(U_2 U_2')$ . Define the estimator of the variance  $\sigma_r^2$  as

$$\widehat{\sigma}_r^2 = \text{vec}(\widehat{U}_2 \widehat{U}_2')' D_n \widehat{C} D_n' \text{vec}(\widehat{U}_2 \widehat{U}_2'), \quad (4.16)$$

where  $\widehat{C}$  is defined by (2.11) and  $n \times (n-r)$  matrix  $\widehat{U}_2$  enters into the partition of an orthogonal matrix  $\widehat{U} = (\widehat{U}_1 \widehat{U}_2)$  such that  $\widehat{U}' \widehat{M} \widehat{U} = \text{diag}(\widehat{v}_n, \dots, \widehat{v}_1)$ . Setting

$$\widehat{\xi}_{\text{eig}}(r) = \widehat{\sigma}_r^{-1} \widehat{\eta}_{\text{eig}}(r), \quad (4.17)$$

we obtain the following result.

**Theorem 4.4** *Suppose that Assumptions (A\*) and (B) hold, and let  $\widehat{\xi}_{\text{eig}}(r)$  be defined by (4.17). Then, under  $H_0 : \text{rk}\{M\} = r$ , we have*

$$\widehat{\xi}_{\text{eig}}(r) \xrightarrow{d} \mathcal{N}(0, 1). \quad (4.18)$$

Under  $H_1 : \text{rk}\{M\} > r$ , we have  $\widehat{\xi}_{\text{eig}}(r) \rightarrow_p +\infty$ .

## 5 MINCHI2 rank test

Another way to test for  $\text{rk}\{M\}$  under Assumptions (A) is to consider the test statistic defined as the sum of the squared eigenvalues. In this case, Assumption (B) may be dropped. Consider the ordered eigenvalues  $\widehat{\gamma}_1^2 \leq \dots \leq \widehat{\gamma}_n^2$  of the matrix  $(\widehat{M} \widehat{F}' \widehat{F})^2 = \widehat{M} \widehat{F}' \widehat{F} \widehat{M} \widehat{F}' \widehat{F}$ . (With the notation of Remark 4.1, we have  $\widehat{\gamma}_k^2 = \widehat{\lambda}_l^2$  for some indices  $k$  and  $l$ .) Let

$$\widehat{\xi}_{\text{mcs}}(r) = N \sum_{k=1}^{n-r} \widehat{\gamma}_k^2, \quad r = 0, \dots, n. \quad (5.1)$$

The next result, proved in Appendix A, follows easily from Theorem 4.1.

**Theorem 5.1** *Suppose that Assumptions (A) hold and denote  $\text{rk}\{M\} = q$ . Then, under  $H_0 : \text{rk}\{M\} \leq r$ , we have*

$$\widehat{\xi}_{\text{mcs}}(r) \xrightarrow{d} \sum_{k=1}^{n-r} (\gamma_k(\mathcal{Z}_{n-q}))^2 \stackrel{d}{\leq} \chi^2((n-r)(n-r+1)/2), \quad (5.2)$$

where  $\leq_d$  becomes  $=_d$  in the case  $r = q$ . Under  $H_1 : \text{rk}\{M\} > r$ , we have  $\widehat{\xi}_{\text{mcs}}(r) \rightarrow_p +\infty$ .

We use  $mcs$  in the subindex of the statistic (5.1) because the statistic  $\widehat{\xi}_{mcs}$  can, in fact, be viewed as the MINCHI2 statistic considered, in particular, by Cragg and Donald (1997), Robin and Smith (2000).

**Proposition 5.1** *Suppose that Assumptions (A) hold and  $\widehat{\xi}_{mcs}(r)$  is defined by (5.1). Then, for  $r = 0, \dots, n$ ,*

$$\widehat{\xi}_{mcs}(r) = N \min_{\text{rk}\{\mathcal{M}\} \leq r} \text{vec}(\widehat{M} - \mathcal{M})' (\widehat{F}' \widehat{F} \otimes \widehat{F}' \widehat{F}) \text{vec}(\widehat{M} - \mathcal{M}). \quad (5.3)$$

Proposition 5.1 is proved in Appendix A. We expect (though do not have a proof) that the minimum over  $\text{rk}\{\mathcal{M}\} \leq r$  in (5.3) can be replaced by the minimum over  $\text{rk}\{\mathcal{M}\} \leq r$ ,  $\mathcal{M} = \mathcal{M}'$ . In this case, the statistic (5.3) can be written as

$$\widehat{\xi}_{mcs}(r) = N \min_{\text{rk}\{\mathcal{M}\} \leq r, \mathcal{M} = \mathcal{M}'} \text{vech}(\widehat{M} - \mathcal{M})' \widehat{C}^{-1} \text{vech}(\widehat{M} - \mathcal{M}), \quad (5.4)$$

where  $\widehat{C} = (D'_n (\widehat{F}' \widehat{F} \otimes \widehat{F}' \widehat{F}) D_n)^{-1}$  is a consistent estimator of the covariance matrix  $C$  appearing in (2.9).

The expression (5.4) is that of the MINCHI2 statistic commonly used for symmetric matrices under more general Assumptions ( $A^*$ ). See, for example, Cragg and Donald (1997), p. 235. The asymptotics of this statistic are given in the following result.

**Theorem 5.2** *Suppose that Assumptions ( $A^*$ ) hold. Let  $\widehat{\xi}_{mcs}(r)$  be defined by (5.4). Then, under  $H_0 : \text{rk}\{M\} = r$ , we have*

$$\widehat{\xi}_{mcs}(r) \xrightarrow{d} \chi^2((n-r)(n-r+1)/2) \quad (5.5)$$

and, under  $H_1 : \text{rk}\{M\} > r$ , we have  $\widehat{\xi}_{mcs}(r) \rightarrow_p +\infty$ .

For completeness, we provide a standard proof of Theorem 5.2 in Appendix A.

**Remark 5.1** Suppose that Assumptions ( $A^*$ ) hold. Robin and Smith (1995), Section 3.6 (applied to symmetric matrices), consider the MINCHI2 statistic defined as

$$\widehat{\xi}_{mcs}(r) = N \min_{\text{rk}\{\mathcal{M}\} \leq r, \mathcal{M} = \mathcal{M}'} \text{vec}(\widehat{M} - \mathcal{M})' \widehat{W}^- \text{vec}(\widehat{M} - \mathcal{M}), \quad (5.6)$$

where  $\widehat{W} = D_n \widehat{C} D'_n$  appears in (2.13) and  $(\cdot)^-$  denotes a generalized reflexive inverse of a matrix. Robin and Smith, moreover, write  $\widehat{C} = (D'_n \widehat{U}^{-1} D_n)^{-1}$  for some invertible matrix  $\widehat{U}$  and suggest using the inverse

$$\widehat{W}^- = \widehat{U}^{-1} D_n (D'_n \widehat{U}^{-1} D_n)^{-1} D'_n \widehat{U}^{-1}.$$

They show that the statistic (5.6) has the asymptotic properties described in Theorem 5.2. Given Theorem 5.2, this is not surprising since  $\text{vec}(\widehat{M} - \mathcal{M}) = D_n \text{vech}(\widehat{M} - \mathcal{M})$  and  $D'_n \widehat{W}^- D_n = D'_n \widehat{U}^{-1} D_n = \widehat{C}^{-1}$  so that expression (5.6) is identical to (5.3).

Though the MINCHI2 statistic is appealing theoretically, it is notorious for problems associated with numerical optimization. When  $C$  does not have a Kronecker product structure, numerical optimization procedures to compute (5.4) sometimes do not converge to a global minimum.

## 6 LDU rank test

The LDU rank test for  $H_0 : \text{rk}\{M\} = r$  with an  $n \times k$  unrestricted matrix  $M$  was introduced by Gill and Lewbel (1992), and subsequently corrected by Cragg and Donald (1996). (Another name for the LDU rank test is the Gaussian elimination rank test.) The test uses the Gaussian elimination procedure with complete pivoting (Golub and Van Loan (1996)) to transform  $M$  and its estimator  $\widehat{M}$  into

$$M(r) = \begin{pmatrix} M_{11}(r) & M_{12}(r) \\ 0 & M_{22}(r) \end{pmatrix} = (A_r P_r) \dots (A_1 P_1) M Q_1 \dots Q_r, \quad (6.1)$$

and analogous expression with the hats for  $\widehat{M}$ . Here,  $P_j, Q_j$  are the permutation matrices corresponding to complete pivoting, and  $A_j$  account for the Gaussian elimination steps. One has that  $\text{rk}\{M\} = r$  if and only if  $M_{22}(r) = 0$ . Hence,  $H_0 : \text{rk}\{M\} = r$  is equivalent to  $H_0 : M_{22}(r) = 0$ .

Supposing for simplicity that  $P_j = I_n$  and  $Q_j = I_n$ , Cragg and Donald (1996) showed that the test statistic

$$\widehat{\xi}_{ldu}(r) = N \text{vec}(\widehat{M}_{22}(r))' (\widehat{\Pi} \widehat{W} \widehat{\Pi}')^{-1} \text{vec}(\widehat{M}_{22}(r)) \quad (6.2)$$

has a limiting  $\chi^2((n-r)(k-r))$  distribution under  $H_0 : \text{rk}\{M\} = r$ , and that  $\widehat{\xi}_{ldu}(r) \rightarrow_p +\infty$  under the alternative. The matrix  $\widehat{W}$  in (6.2) is a consistent estimator for the limiting covariance matrix  $W$  of  $\sqrt{N}(\widehat{M} - M)$ . The matrix  $\widehat{\Pi}$  is defined as

$$\widehat{\Pi} = \begin{pmatrix} -\widehat{M}_{21} \widehat{M}_{11}^{-1} & I_{n-r} \end{pmatrix} \otimes \begin{pmatrix} -\widehat{M}'_{12} (\widehat{M}'_{11})^{-1} & I_{k-r} \end{pmatrix}, \quad (6.3)$$

where  $\widehat{M} = (\widehat{M}_{11} \widehat{M}_{12}; \widehat{M}_{21} \widehat{M}_{22})$  is the partition of the matrix  $\widehat{M}$  with an  $r \times r$  submatrix  $\widehat{M}_{11}$ . If  $P_j$  and  $Q_j$  are not identity matrices, the matrix  $\widehat{M}$  can be permuted in advance so that complete pivoting becomes unnecessary. This, however, requires properly adjusting  $\widehat{\Pi} \widehat{W} \widehat{\Pi}'$  appearing in (6.2).

When  $M$  and  $\widehat{M}$  are symmetric, Robin and Smith (1995) suggest to replace the inverse of the matrix  $\widehat{\Pi} \widehat{W} \widehat{\Pi}'$  in (2.13) by its generalized reflexive inverse. These authors state that thus obtained statistic has a limiting  $\chi^2((n-r)(n-r+1)/2)$  distribution under the null, and diverges under the alternative. This approach is also used in Kapetanios and Camba-Mendez (1999), Ratsimalahelo (2002) and Robin and Smith (2000).

One may expect that using the generalized inverse may be avoided if  $\text{vec}(\widehat{M}_{22}(r))$  in (2.13) is replaced by  $\text{vech}(\widehat{M}_{22}(r))$ . Observe, however, that  $\text{vech}(\widehat{M}_{22}(r))$  may not even be defined because complete pivoting does not preserve symmetry and hence  $\widehat{M}_{22}(r)$  may not be symmetric. The matrix  $\widehat{M}_{22}(r)$  may be kept symmetric by working with a pivoting method which preserves symmetry. This avoids using generalized inverses and simplifies the proofs. One such method, attributed to William Kahan (see pp. 646–647 in Bunch and Parlett (1971)), is the following. Let  $M = (m_{ij})$  be an  $n \times n$  symmetric matrix.

**Gaussian elimination with symmetric pivoting: Step 1:** Search the entire matrix  $M$  for  $\max\{m_{ii}^2, |m_{jj}m_{kk} - m_{jk}^2|, i, j, k\}$ . If  $m_{ii}^2$  is the maximum, set  $S = m_{ii}$ . If the maximum is  $|m_{jj}m_{kk} - m_{jk}^2|$ , then set  $S = (m_{jj} m_{jk}; m_{kj} m_{kk})$ . **Step 2:** Permute rows and columns to bring the  $1 \times 1$  or  $2 \times 2$  pivot  $S$  into the upper-left corner of the matrix  $M$ . Observe that these permutations preserve symmetry. **Step 3:** Use the Gaussian elimination to transform  $M$  into

$$\begin{pmatrix} m_{11}(1) & m_{12}(1) \\ 0 & m_{22}(1) \end{pmatrix} = (B_1 R_1) M R'_1,$$

where  $R_1$  and  $B_1$  correspond to the permutation and the Gaussian elimination steps, respectively. Here,  $m_{11}(1) = S$  and  $m_{22}(1)$  is a symmetric matrix defined by  $m_{22}(1) = U - TS^{-1}T'$ , where  $R_1MR'_1 = (ST'; TU)$  is a partition of the permuted matrix. **Step 4:** Apply the previous steps to  $m_{22}(1)$  and so on to obtain the expression

$$\begin{pmatrix} m_{11}(k) & m_{12}(k) \\ 0 & m_{22}(k) \end{pmatrix} = (B_k R_k) \dots (B_1 R_1) M R'_1 \dots R'_k, \quad (6.4)$$

where  $m_{22}(k)$  is symmetric.

Applying the Gaussian elimination with symmetric pivoting to the matrix  $\widehat{M}$ , we obtain

$$\begin{pmatrix} \widehat{m}_{11}(k) & \widehat{m}_{12}(k) \\ 0 & \widehat{m}_{22}(k) \end{pmatrix} = (\widehat{B}_k \widehat{R}_k) \dots (\widehat{B}_1 \widehat{R}_1) \widehat{M} \widehat{R}'_1 \dots \widehat{R}'_k. \quad (6.5)$$

By permuting the matrix  $\widehat{M}$  beforehand, we may suppose that symmetric permutations become no longer necessary (that is,  $\widehat{R}_k = I_n$  above).

To track  $\dim(m_{22}(k))$ , it is necessary to introduce the integers  $k(r) = \min\{k : \dim(m_{22}(k)) \leq n-r\}$  and  $l(r) = \dim(m_{11}(k(r)))$ . Note that  $l(r) = r$  or  $r+1$ . Observe also that, if  $\text{rk}\{M\} < n$ , then the Gaussian elimination procedure with symmetric pivoting can be applied till  $m_{22}(k_0) = 0$  for some  $k_0$ . In fact,  $k_0 = k(\text{rk}\{M\})$ ,  $\dim(m_{22}(k_0)) = n - \text{rk}\{M\}$  and  $l(\text{rk}\{M\}) = \text{rk}\{M\}$ . Let also  $\widehat{k}(r)$  and  $\widehat{l}(r)$  be defined similarly by using  $\widehat{M}$ .

Consider the symmetric analogue of the LDU statistic

$$\widehat{\xi}_{sldu}(r) = N \text{vech}(\widehat{m}_{22}(\widehat{k}(r)))' (\widehat{\Pi}_s \widehat{C} \widehat{\Pi}'_s)^{-1} \text{vech}(\widehat{m}_{22}(\widehat{k}(r))), \quad (6.6)$$

where

$$\widehat{\Pi}_s = D_{n-\widehat{l}(r)}^+ (\widehat{\Phi} \otimes \widehat{\Phi}) D_n \quad \text{with} \quad \widehat{\Phi} = \begin{pmatrix} -\widehat{M}_{21} \widehat{M}_{11}^{-1} & I_{n-\widehat{l}(r)} \end{pmatrix} \quad (6.7)$$

and  $\widehat{M} = (\widehat{M}_{11} \widehat{M}_{12}; \widehat{M}_{21} \widehat{M}_{22})$  is a partition with  $\widehat{l}(r) \times \widehat{l}(r)$  submatrix  $\widehat{M}_{11}$ .

**Theorem 6.1** *Suppose that Assumptions (A\*) hold. Let  $\widehat{\xi}_{sldu}(r)$  be defined by (6.6). Then, under  $H_0 : \text{rk}\{M\} = r$ ,  $\widehat{\xi}_{sldu}(r) \rightarrow_d \chi^2((n-r)(n-r+1)/2)$  and, under  $H_1 : \text{rk}\{M\} > r$ ,  $\widehat{\xi}_{sldu}(r) \rightarrow_p +\infty$ .*

This theorem is proved in Appendix A. We show in the proof of Theorem 6.1 that  $\widehat{\xi}_{sldu}(r) - \widehat{\xi}_{mcs}(r) \rightarrow_p 0$ , where

$$\widehat{\xi}_{mcs}(r) = N \min_{\text{rk}\{\mathcal{M}\}=r} \text{vec}(\widehat{M} - \mathcal{M})' D_n^+ \widehat{C}^{-1} D_n^+ \text{vec}(\widehat{M} - \mathcal{M}). \quad (6.8)$$

When the minimum is over  $\text{rk}\{\mathcal{M}\} \leq r$ ,  $\mathcal{M} = \mathcal{M}'$  in (6.8), we can write  $\text{vec}(\widehat{M} - \mathcal{M}) = D_n \text{vech}(\widehat{M} - \mathcal{M})$ . Since  $D_n^+ D_n = I_{n(n+1)/2}$ , the statistic (6.8) becomes that in (5.4). Since  $\widehat{M}$  is symmetric, we expect (though do not have a proof) that the two statistics are, in fact, the same.

Note that, since Gaussian elimination with symmetric pivoting may involve  $2 \times 2$  pivots, we may have  $\widehat{\xi}_{sldu}(r) = \widehat{\xi}_{sldu}(r+1)$  for some  $r$ . Note also that (6.6) is not properly defined in the following situation. If  $\widehat{l}(n-2) = n-2$ , the remaining symmetric matrix  $\widehat{m}_{22}(\widehat{k}(n-2))$  is  $2 \times 2$ . If  $\widehat{m}_{22}(\widehat{k}(n-2))$  is the chosen  $2 \times 2$  pivot in the next Gaussian elimination step, the elimination method cannot continue and hence  $\widehat{\xi}_{sldu}(n-1)$  is not well defined by (6.6). In this case, we set  $\widehat{\xi}_{sldu}(n-1) = \widehat{\xi}_{sldu}(n-2)$ .

**Remark 6.1** The applied mathematics literature offers a number of other Gaussian elimination like methods for symmetric matrices which preserve symmetry after each elimination step. The best known method is perhaps the diagonal pivoting method and its various modifications (see Bunch (1971), Bunch and Parlett (1971), Bunch and Kaufman (1977), and others). We did not use the diagonal method above because it only applies to symmetric nonsingular matrices. In other words, even though the method would apply to  $\widehat{M}$  where typically  $\text{rk}\{\widehat{M}\} = n$ , it would not apply to  $M$  supposing  $\text{rk}\{M\} < n$ . Hence, we would not be able to establish the asymptotic results analogous to those in Theorem 6.1.

## 7 SVD rank test

Kleiberger and Paap (2003) have recently proposed an ingenious rank test based on Singular Value Decomposition (SVD) of a matrix. We shall adapt here their test to symmetric matrices under Assumptions ( $A^*$ ). The SVD of a symmetric  $n \times n$  matrix  $M$  is given by

$$M = USV', \quad (7.1)$$

where matrices  $U$  ( $n \times n$ ) and  $V$  ( $n \times n$ ) are orthogonal, and a diagonal matrix  $S$  ( $n \times n$ ) consists of decreasing singular values of  $M$  on its diagonal. For symmetric matrices, the singular values are just the absolute values of their eigenvalues.

For  $r = 0, \dots, n$ , partition (7.1) as

$$M = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} \begin{pmatrix} V'_{11} & V'_{21} \\ V'_{12} & V'_{22} \end{pmatrix}, \quad (7.2)$$

where matrices  $U_{11}$ ,  $S_1$  and  $V_{11}$  are  $r \times r$  and the dimensions of other submatrices can be deduced. Following Kleiberger and Paap (2003), define the matrices

$$A_r = \begin{pmatrix} U_{11} \\ U_{21} \end{pmatrix} S_1 V'_{11}, \quad B_r = (I_r \quad (V'_{11})^{-1} V'_{21}), \quad (7.3)$$

$$A_{r,\perp} = \begin{pmatrix} U_{12} \\ U_{22} \end{pmatrix} U_{22}^{-1} (U_{22} U'_{22})^{1/2}, \quad B_{r,\perp} = (V_{22} V'_{22})^{1/2} V'^{-1}_{22} (V'_{12} \quad V'_{22}) \quad (7.4)$$

and

$$\Lambda_r = (U_{22} U'_{22})^{-1/2} U_{22} S_2 V'_{22} (V_{22} V'_{22})^{-1/2}. \quad (7.5)$$

Then,

$$M = A_r B_r + A_{r,\perp} \Lambda_r B_{r,\perp}, \quad (7.6)$$

and  $A'_r A_{r,\perp} = 0$ ,  $B_{r,\perp} B'_r = 0$ ,  $A'_{r,\perp} A_r = I_{n-r}$ ,  $B_{r,\perp} B'_{r,\perp} = I_{n-r}$ . The null hypothesis  $H_0 : \text{rk}\{M\} = r$  is equivalent to  $H_0 : \Lambda_r = 0$ . The above also applies to the matrix  $\widehat{M}$  in which case the corresponding matrices in (7.1)-(7.6) will be denoted with the hats.

Observe that  $\widehat{\Lambda}_r$  is defined as orthogonal transformation of the singular values  $\widehat{S}_2$  of the matrix  $\widehat{M}$  which are absolute values of its eigenvalues. We have established the asymptotics of the eigenvalues in Remark 4.1. In contrast to that result, the orthogonal transformation of the eigenvalues is, in fact, asymptotically Gaussian. This result is analogous to Theorem 1 in Kleiberger and Paap (2003). It also shows that  $\widehat{\Lambda}_r$  is symmetric, and that  $\widehat{A}_{r,\perp}$  and  $\widehat{B}'_{r,\perp}$  are easily related.

**Proposition 7.1** *With the above notation,  $\widehat{\Lambda}_r, \Lambda_r$  are symmetric and  $A_{r,\perp} = B'_{r,\perp}$ ,  $\widehat{A}_{r,\perp} = \widehat{B}'_{r,\perp}$ . Moreover, under  $H_0 : \text{rk}\{M\} = r$  and Assumptions  $(A^*)$ ,*

$$\sqrt{N}\text{vech}(\widehat{\Lambda}_r) \xrightarrow{d} \mathcal{N}(0, \Omega_r), \quad (7.7)$$

where

$$\Omega_r = D_{n-r}^+(B_{r,\perp} \otimes A'_{r,\perp})D_n C D'_n (B'_{r,\perp} \otimes A_{r,\perp})D_{n-r}^{+ \prime}. \quad (7.8)$$

Let  $\widehat{\Omega}_r$  be defined through (7.8) with  $\widehat{A}_{r,\perp}$  and  $\widehat{C}$ . Based on Proposition 7.1, consider the test statistic

$$\widehat{\xi}_{svd}(r) = N\text{vech}(\widehat{\Lambda}_r)' \widehat{\Omega}_r^{-1} \text{vech}(\widehat{\Lambda}_r). \quad (7.9)$$

The following result is a direct consequence of Proposition 7.1.

**Theorem 7.1** *Suppose that Assumptions  $(A^*)$  hold and that the matrix  $\Omega_r$  is nonsingular. Then, under  $H_0 : \text{rk}\{M\} = r$ ,*

$$\widehat{\xi}_{svd}(r) \xrightarrow{d} \chi^2((n-r)(n-r+1)/2) \quad (7.10)$$

and under  $H_1 : \text{rk}\{M\} > r$ ,  $\widehat{\xi}_{svd}(r) \rightarrow_p +\infty$ .

As in Kleibergen and Paap (2003), under stronger Assumptions  $(A)$ , one can relate the SVD and MINCHI2 rank tests. The SVD statistic  $\widehat{\xi}_{svd}$  for the scaled matrix

$$\widehat{M}^* = \widehat{F} \widehat{M} \widehat{F}'$$

and the corresponding covariance matrix

$$\widehat{C}^* = D_n^+(\widehat{F} \otimes \widehat{F})D_n \widehat{C} D'_n (\widehat{F}' \otimes \widehat{F}')D_n^{+ \prime}$$

satisfies

$$\widehat{\xi}_{svd}(r) = \widehat{\xi}_{mcs}(r), \quad r = 0, \dots, n, \quad (7.11)$$

where  $\widehat{\xi}_{mcs}(r)$  is the MINCHI2 statistic defined by (5.1).

## 8 Simulation study

We examine here the small sample properties of the considered rank tests through a small simulation study. The set up is that of Example 3.2. We generate  $Y_i = BZ_i + \epsilon_i$ ,  $i = 1, \dots, N$ , where  $B$  is a fixed, symmetric, positive semidefinite,  $6 \times 6$  matrix with  $\text{rk}\{B\} = 3$ . (The eigenvalues of  $B$  are 0, 0, 0, 3.51, 4.79 and 24.69.) The variables  $Z_i$  are i.i.d.  $\mathcal{U}[0, 1]$ . The error terms  $\epsilon_i = Hu_i$  with i.i.d.  $\mathcal{U}[-1, 1]$  variables  $u_i$  and a fixed, nonsingular matrix  $H$ . The sample size  $N = 200, 500$  or  $1000$ .

For each sample size, we generate 2000 replications of  $(Y_i, Z_i)$ . For each replication, we estimate  $B$  through least squares (3.12), and the related covariance matrix  $C$  based on (3.13). Having  $\widehat{B}$  and  $\widehat{C}$ , we test for  $H_0 : \text{rk}\{B\} = r$ ,  $r = 0, 1, \dots, 5$ , under Assumptions  $(A^*)$  by using either the EIG, LDU or SVD rank test, with the corresponding statistic  $\widehat{\xi}(r)$ . (The MINCHI2 test is not considered for practical problems associated with optimization.)

In Figure 1–3 at the end of the paper, for different sample sizes  $N$  and rank values  $r$ , we present the PP-plots of a probability  $p \in (0, 1)$  on the vertical axis against  $\alpha_r(p) = P(\widehat{\xi}(r) > c_r(p))$  on the horizontal axis, estimated from 2000 replications of the corresponding statistic  $\widehat{\xi}(r)$ . Here,  $c_r(p)$  is the

nominal critical value for the limiting distribution  $\xi(r)$  associated with the statistic  $\widehat{\xi}(r)$ , that is,  $c_r(p)$  is such that  $P(\xi(r) > c_r(p)) = 1 - p$ . If  $\widehat{\xi}(r)$  has the limiting distribution  $\xi(r)$ , then  $\alpha_r(p) \equiv p$ . If  $\widehat{\xi}(r)$  is skewed to the right of the limiting distribution  $\xi(r)$ , then  $\alpha_r(p) > p$ . Analogous PP-plots were considered by Robin and Smith (2000).

The PP-plots for  $r = 3$  (true rank of  $B$ ) in Figures 1–3 suggest that the EIG test is undersized, and that the LDU and SVD tests are oversized. This becomes less pronounced as the sample size increases. In other words, for smaller sample sizes, the EIG test is likely to accept too low a rank while the LDU and SVD tests are likely to accept too high a rank. An idea of the power properties of tests can be seen from the PP-plots for  $r = 0, 1, 2$ . The LDU and SVD tests appear to have better powers than the EIG test. Observe, however, that these powers are not size-adjusted. In view of the discussion on sizes, adjusting would improve the power of the EIG test and lower those of the LDU and SVD tests. The PP-plots for  $r = 4, 5$  illustrate nonstandard distributions of the corresponding statistics  $\widehat{\xi}(r)$  when  $r > \text{rk}\{B\}$ .

Observe also that the PP-plots for  $r = 3$  and the SVD test are surprisingly close to the asymptotic  $45^\circ$  line even for small samples. Deviations from the straight line occur, however, at the key right tail of the distribution. Observe also from the PP-plots for  $r = 3$  that, as the sample size increases, all the statistics appear to converge to the limiting distribution, with the SVD statistic converging faster. (The PP-plots for  $r = 3$  and the EIG, LDU tests become a straight line when  $N = 10,000$ .) This convergence confirms the theoretical limit results for the test statistic established in the paper. (Note also that the analogous plots in Figures 2, 4 and 6 of Robin and Smith (2000) do not appear to confirm the limiting results for tests involving LUCP, that is, complete but nonsymmetric pivoting.)

Finally, let us note that we implemented the considered rank tests in MATLAB. The written codes are available from the authors upon request. Though irrelevant for small sample sizes and small simulation studies, we also note that the EIG test is faster than the SVD test which is faster than the LDU test.

## 9 Application

We apply here the considered tests to estimate the rank of a demand system. The set up is that of Example 3.1. The demand system is assumed to have a stochastic form  $Y_i = F(X_i) + \epsilon_i$ ,  $i = 1, \dots, N$ , where  $Y_i$  and  $X_i$  are the vector of the shares of goods and the income corresponding to the  $i$ th consumer, respectively, and  $\epsilon_i$  is the noise term with either nonsingular  $E\epsilon_i\epsilon_i' = \Sigma$  (homoscedastic assumptions) or  $E(\epsilon_i\epsilon_i'|X_i) = \Sigma(X_i)$  (heteroskedastic assumptions). Because all budget shares add up to 1 and the assumed covariance matrix of perturbation terms is nonsingular, one budget share is, in fact, eliminated from the vector  $Y_i$ . We are interested in estimating  $\text{rk}\{F\}$  with  $x^1 \equiv 1$  (Example 3.1 and Donald (1997)). This can be done by estimating the rank of the corresponding symmetric, positive semidefinite matrix  $\Gamma_w$  as in Example 3.1, under either (3.6) (Assumptions (A); homoscedasticity assumptions) or (3.7) (Assumptions (A\*); heteroskedasticity assumptions). The rank of the full (original) demand system is deduced by adding 1 to the estimated rank of  $\Gamma_w$  (Donald (1997)).

The data set of a demand system used contains information on expenditures by  $N = 897$  consumers across the U.S. taken from the U.S. CEX survey of the first quarter of 2000.<sup>1</sup> Households considered have the following homogeneous characteristics: married couples, renters or homeowners with or without mortgage, age of the head between 25 and 60, and total income  $e^{X_i}$  larger than or equal to \$3,000

<sup>1</sup>U.S. Dep. of Labor, Bureau of Labor Statistics. Consumer Expenditure Survey, 1999: Interview Survey and Detailed Expenditure Files [Computer file]. Washington, DC: U.S. Dept. of Labor, Bureau of Labor Statistics [producer], 2001. Ann Arbor, MI: Inter-University Consortium for Political and Social Research [distributor], 2001.

but smaller than or equal to \$75,000. The vector  $Y_i$  contains budget shares of 5 categories of goods (food, health care, transportation, household and apparel (clothing); the share of miscellaneous goods was dropped). This data set (with additional price information) was also used in Fortuna (2004).

Tables 1 and 2 at the back of the paper, present rank estimation results for the full demand system. For kernel smoothing, we use the Epanechnikov kernel  $K$  of order 2. The smoothing parameter  $h$  takes one of the values 0.1, 0.3, 0.45 and 0.55. These choices were suggested in part by the fact that  $h = 0.45$  is the optimal smoothing parameter obtained by the generalization cross validation procedure for the data set consisting of all expenditure shares. The entries in Tables 1 and 2 are the  $p$ -values for the rank tests applied to the constructed data set. The results suggest that the rank of the full demand system is 3 under homoscedastic and heteroskedastic assumptions for each of the rank tests considered. Rank 3 has also been found for other demand systems in Lewbel (1991), Donald (1997). Observe that the  $p$ -values for the MINCHI2 and SVD tests are the same in Table 1, confirming the equality of the corresponding statistics in (7.11). The  $p$ -values for the MINCHI2 test are not reported in Table 2 because of convergence problems in optimization. In both tables and for larger  $h = 0.3, 0.45, 0.55$ , the  $p$ -values for the EIG test are bigger than those for the LDU and SVD tests. This confirms the results obtained in the previous section that the EIG test is likely to accept lower rank than the other tests.

## A Technical proofs

PROOF OF PROPOSITION 2.2: If (2.14) holds, then  $\sqrt{N}F(\widehat{M} - M)G' \rightarrow_d \mathcal{Z}_n$  and hence  $\sqrt{N}(\widehat{M} - M) \rightarrow_d F^{-1}\mathcal{Z}_n(G')^{-1}$ . Since  $\widehat{M} - M$  is symmetric, the matrix  $F^{-1}\mathcal{Z}_n(G')^{-1}$  is symmetric, that is,  $F^{-1}\mathcal{Z}_n(G')^{-1} = G^{-1}\mathcal{Z}_n(F^{-1})'$ . Then,  $\mathcal{Z}_n A' = A\mathcal{Z}_n$  with  $A = FG^{-1}$  or  $(A \otimes I_n)\text{vec}(\mathcal{Z}_n) = (I_n \otimes A)\text{vec}(\mathcal{Z}_n)$  or  $(A \otimes I_n)D_n\text{vech}(\mathcal{Z}_n) = (I_n \otimes A)D_n\text{vech}(\mathcal{Z}_n)$ , where  $D_n$  is defined by (2.4). Since  $\text{vech}(\mathcal{Z}_n)$  consists of independent normal random variables, we have

$$((A \otimes I_n) - (I_n \otimes A))D_n = 0. \quad (\text{A.1})$$

It is enough to show that (A.1) implies  $A = \text{const } I_n$  (then  $FG^{-1} = \text{const } I_n$  and hence  $F = \text{const } G$ ). Observe that

$$(A \otimes I_n) - (I_n \otimes A) = \begin{pmatrix} a_{11}I_n - A & a_{12}I_n & \dots & a_{1n}I_n \\ & A^* & & \end{pmatrix} \quad (\text{A.2})$$

and also that, for  $n \geq 3$ ,

$$D_n = \begin{pmatrix} I_n & 0_n \\ X_2 & Y_2 \\ \vdots & \vdots & D_n^* \\ X_n & Y_n \end{pmatrix} \quad (\text{A.3})$$

where the  $n \times n$  matrix  $X_k$  is such that its first row is  $(0 \dots 1 \dots 0)$  with 1 at the  $k$ th position and the other entries are zero,

$$Y_2 = \begin{pmatrix} 0_{1 \times (n-1)} & 0 \\ I_{n-1} & 0_{(n-1) \times 1} \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 1 \\ & 0_{(n-3) \times n} & & \end{pmatrix}$$

and  $Y_k$ ,  $k \geq 4$ , is such that its second row is  $(0 \dots 1 \dots 0)$  with 1 at at the  $(k-1)$ th position and the other entries are zero. Relations (A.1), (A.2) and (A.3) then imply that, when  $n \geq 3$ ,

$$\begin{aligned} (a_{11}I_n - A) + a_{12}X_2 + \dots + a_{1n}X_n &= 0, \\ a_{12}Y_2 + a_{13}Y_3 + \dots + a_{1n}Y_n &= 0. \end{aligned}$$

By using the expressions for  $X_k$ 's and  $Y_k$ 's, one can see that these equations imply  $a_{11} = a_{22} = \dots = a_{nn}$  and  $a_{ij} = 0$ ,  $i \neq j$ , that is,  $A = \text{const } I_n$ . The cases  $n = 1, 2$  can be dealt with directly.  $\square$

**PROOF OF PROPOSITION 2.3:** Consider the case of positive semidefiniteness. We shall argue by contradiction. Suppose first that  $M$  is a diagonal matrix  $M = \text{diag}(m_{11}, \dots, m_{nn})$ . Since  $\text{rk}\{M\} < n$ , we may suppose without loss of generality that  $m_{nn} = 0$ . Since  $\widehat{M} = (\widehat{m}_{ij})$  is asymptotically normal and  $m_{nn} = 0$ , we obtain that  $\sqrt{N}\widehat{m}_{nn} \rightarrow_d \mathcal{N}(0, \sigma_n^2)$ . On the other hand, since  $\widehat{M}$  is positive semidefinite,  $\widehat{m}_{nn} = u' \widehat{M} u \geq 0$ , with  $u' = (0 \dots 0 1)$ . This implies that  $\sigma_n^2 = 0$ , and hence that the covariance matrix  $C$  is singular (the row of  $C$  corresponding to  $\sqrt{N}\widehat{m}_{nn}$  consists of zeros).

Consider now the case when  $M$  is any symmetric matrix. There is an orthogonal transformation  $U$  such that  $UMU' = M^* = \text{diag}(m_{11}^*, \dots, m_{nn}^*)$  is diagonal. Let  $\widehat{M}^* = U\widehat{M}U'$  and observe that  $\widehat{M}^*$  is also positive semidefinite. Moreover, we have that

$$\sqrt{N}(\text{vech}(\widehat{M}^*) - \text{vech}(M^*)) \xrightarrow{d} \mathcal{N}(0, D_n^+(U' \otimes U')D_n C D_n'(U \otimes U)D_n^{+'}). \quad (\text{A.4})$$

The matrix  $D_n^+(U' \otimes U')D_n$  is positive definite by Theorem 13, (c), in Magnus and Neudecker (1999), pp. 49-50. Hence, the covariance matrix in (A.4) is also positive definite and the contradiction follows as in the simple case considered above.  $\square$

**PROOF OF THEOREM 4.1:** We adapt here the proof of Theorem 3.1 in Robin and Smith (2000). Let  $c_i$ ,  $i = 1, \dots, n$ , be the eigenvectors of the matrix  $F'FM$  corresponding to the eigenvalue  $\lambda_i$ , that is,  $F'FMc_i = \lambda_i c_i$ . Denote  $C = (c_1 \dots c_n) = (C_{n-r} \ C_r)$  with  $C_{n-r} = (c_1 \dots c_{n-r})$  and  $C_r = (c_{n-r+1} \dots c_n)$ . (This  $C$  is used only in this proof and should not be confused with the covariance-like matrix  $C$  appearing in (2.10).) We may normalize  $C$  as  $C'(F'F)^{-1}C = I_n$ . Let also  $\Lambda_r = \text{diag}(\lambda_{n-r+1}, \dots, \lambda_n)$  and  $C'^{-1} = (C'^{n-r} \ C^r)$ . Observe that  $C_r' C^r = I_r$  and  $C_{n-r}' C^r = 0$ .

As in Lemma A.1 of Robin and Smith (2000), one can show that

$$M = C^r \Lambda_r C^{r'}. \quad (\text{A.5})$$

By using (A.5), (2.1) and since the eigenvectors in  $C_{n-r}$  correspond to the eigenvalue 0, we have

$$C_r' \widehat{M} = \Lambda_r C^{r'} + O_p(N^{-1/2}), \quad N^{1/2} C_{n-r}' \widehat{M} = C_{n-r}' N^{1/2} (\widehat{M} - M). \quad (\text{A.6})$$

Observe now by (2.2) that

$$0 = \det(\widehat{M} - \widehat{\lambda}_i (\widehat{F}' \widehat{F})^{-1}) = \det(\widehat{M} - \widehat{\lambda}_i (F'F)^{-1}) + o_p(1). \quad (\text{A.7})$$

By using (A.6), since  $\widehat{\lambda}_i \rightarrow_p 0$  for  $i = 1, \dots, n-r$ , and  $C_{n-r}' (F'F)^{-1} C_{n-r} = I_{n-r}$ ,  $C_r' (F'F)^{-1} C_r = I_r$ ,  $C_{n-r}' (F'F)^{-1} C_r = 0$ , we obtain for  $i = 1, \dots, n-r$ ,

$$\begin{aligned} \det(\widehat{M} - \widehat{\lambda}_i (F'F)^{-1}) &= \det \left( \begin{pmatrix} N^{1/4} C_{n-r}' \\ C_r' \end{pmatrix} (\widehat{M} - \widehat{\lambda}_i (F'F)^{-1}) (N^{1/4} C_{n-r} \ C_r) \right) \\ &= \det \left( \begin{pmatrix} N^{1/2} C_{n-r}' (\widehat{M} - M) C_{n-r} & O_p(N^{-1/4}) \\ O_p(N^{-1/4}) & \Lambda_r + O_p(N^{-1/2}) \end{pmatrix} - \widehat{\lambda}_i \begin{pmatrix} N^{1/2} I_{n-r} & 0 \\ 0 & I_r \end{pmatrix} \right) \\ &= \det \left( \begin{pmatrix} N^{1/2} C_{n-r}' (\widehat{M} - M) C_{n-r} & 0 \\ 0 & \Lambda_r \end{pmatrix} - N^{1/2} \widehat{\lambda}_i \begin{pmatrix} I_{n-r} & 0 \\ 0 & 0 \end{pmatrix} \right) + o_p(1) \end{aligned}$$

$$= \det(\Lambda_r) \det(N^{1/2} C'_{n-r} (\widehat{M} - M) C_{n-r} - N^{1/2} \widehat{\lambda}_i I_{n-r}) + o_p(1). \quad (\text{A.8})$$

Hence, by (A.7) and (A.8), for  $i = 1, \dots, n-r$ ,

$$o_p(1) = \det(N^{1/2} C'_{n-r} (\widehat{M} - M) C_{n-r} - N^{1/2} \widehat{\lambda}_i I_{n-r}). \quad (\text{A.9})$$

This shows that  $N^{1/2} \widehat{\lambda}_i$ ,  $i = 1, \dots, n-r$ , are asymptotically the eigenvalues of  $N^{1/2} C'_{n-r} (\widehat{M} - M) C_{n-r}$ . Observe finally that

$$N^{1/2} C'_{n-r} (\widehat{M} - M) C_{n-r} \xrightarrow{d} C'_{n-r} F^{-1} \mathcal{Z}_n F^{-1} C_{n-r} \stackrel{d}{=} \mathcal{Z}_n.$$

The last equality follows since  $\text{vec}(C'_{n-r} F^{-1} \mathcal{Z}_n F^{-1} C_{n-r}) = (C'_{n-r} F^{-1} \otimes C'_{n-r} F^{-1}) \text{vec}(\mathcal{Z}_n) =: W$  and

$$\begin{aligned} EWW' &= (C'_{n-r} F^{-1} \otimes C'_{n-r} F^{-1}) \frac{1}{2} (I_{n^2} + K_n) (F^{-1} C_{n-r} \otimes F^{-1} C_{n-r}) \\ &= (C'_{n-r} F^{-1} \otimes C'_{n-r} F^{-1}) (F^{-1} C_{n-r} \otimes F^{-1} C_{n-r}) \frac{1}{2} (I_{n^2} + K_n) = \frac{1}{2} (I_{n^2} + K_n) = E \text{vec}(\mathcal{Z}_n) \text{vec}(\mathcal{Z}'_n). \end{aligned}$$

The convergence (4.3) follows since  $\widehat{\lambda}_k \rightarrow \lambda_k > 0$ ,  $k = n-r+1, \dots, n$ .  $\square$

PROOF OF THEOREM 4.2: To show (4.5), it is enough to prove the stochastic dominance. This can be done as in the proof of Theorems 1 and 2 in Donald (1997). By the Poincaré separation theorem, we have  $\gamma_j(\mathcal{Z}_{n-q}) \leq \gamma_j(B' \mathcal{Z}_{n-q} B)$  for  $j = 1, \dots, n-r$ , where  $B$  is any  $(n-q) \times (n-r)$  matrix such that  $B'B = I_{n-r}$ . Now take  $B = (0_{(n-r) \times (r-q)} \ I_{n-r})'$  so that  $B'B = I_{n-r}$ . Observe that  $B' \mathcal{Z}_{n-q} B \stackrel{d}{=} \mathcal{Z}_{n-r}$  and hence

$$\frac{1}{\sqrt{n-r}} \sum_{k=1}^{n-r} \gamma_k(\mathcal{Z}_{n-q}) \stackrel{d}{\leq} \frac{1}{\sqrt{n-r}} \sum_{k=1}^{n-r} \gamma_k(\mathcal{Z}_{n-r}) = \frac{1}{\sqrt{n-r}} \text{tr}\{\mathcal{Z}_{n-r}\} \stackrel{d}{=} \mathcal{N}(0, 1).$$

The convergence under  $H_1$  follows from (4.3) in Theorem 4.1.  $\square$

PROOF OF THEOREM 4.3: Arguing as in the proof of Theorem 4.1 above, we may prove that  $\sqrt{N} \widehat{v}_k$ ,  $k = 1, \dots, n-q$ , are asymptotically the ordered eigenvalues of  $U_2' \sqrt{N} (\widehat{M} - M) U_2$ . This shows the convergence (4.12). The proof of (4.13) is straightforward.  $\square$

PROOF OF THEOREM 5.1: The convergence in (5.2) follows from Remark 4.1. To prove the stochastic dominance in (5.2), observe first that

$$\sum_{k=1}^{n-r} (\gamma_k(\mathcal{Z}_{n-q}))^2 = \sum_{k=1}^{n-r} \gamma_k(\mathcal{Z}_{n-q}^2), \quad (\text{A.10})$$

where  $\gamma_k(\mathcal{Z}_{n-q}^2)$ ,  $k = 1, \dots, n-q$ , denote the eigenvalues of  $\mathcal{Z}_{n-q}^2$  in the increasing order. Letting  $B = (0_{(n-r) \times (r-q)} \ I_{n-r})'$ , we can conclude as in the proof of Theorem 4.2 above that

$$\sum_{k=1}^{n-r} \gamma_k(\mathcal{Z}_{n-q}^2) \stackrel{d}{\leq} \sum_{k=1}^{n-r} \gamma_k(B' \mathcal{Z}_{n-q} \mathcal{Z}_{n-q} B) = \sum_{k=1}^{n-r} \gamma_k(\mathcal{Z}_{n-q} B B' \mathcal{Z}_{n-q}) \stackrel{d}{\leq} \sum_{k=1}^{n-r} \gamma_k((B' \mathcal{Z}_{n-q} B)(B' \mathcal{Z}_{n-q} B)). \quad (\text{A.11})$$

Since  $B' \mathcal{Z}_{n-q} B \stackrel{d}{=} \mathcal{Z}_{n-r}$ , it follows from (A.10) and (A.11) that

$$\sum_{k=1}^{n-r} (\gamma_k(\mathcal{Z}_{n-q}))^2 \stackrel{d}{\leq} \sum_{k=1}^{n-r} \gamma_k(\mathcal{Z}_{n-r}^2) = \text{tr}\{\mathcal{Z}_{n-r}^2\} = \text{vec}(\mathcal{Z}_{n-r})' \text{vec}(\mathcal{Z}_{n-r}) \stackrel{d}{=} \chi^2((n-r)(n-r+1)/2)$$

(to see the last equality use the fact  $2(\mathcal{N}(0, 1/2))^2 =_d \mathcal{N}(0, 1)^2$ ).  $\square$

PROOF OF PROPOSITION 5.1: The proof uses some ideas of the proof of Theorem 3 in Cragg and Donald (1993). The restriction  $\text{rk}\{\mathcal{M}\} \leq r$  can be written as  $\Xi\mathcal{M} = 0_{(n-r) \times n}$  or

$$(I_n \otimes \Xi)\text{vec}(\mathcal{M}) = 0_{((n-r)n) \times 1} \quad (\text{A.12})$$

where  $\Xi$  is a  $(n-r) \times n$  matrix with its  $n-r$  rows linearly independent. Moreover, after a proper normalization, we can assume that  $\Xi$  satisfies

$$\Xi(\widehat{F}'\widehat{F})^{-1}\Xi' = I_{n-r}. \quad (\text{A.13})$$

When  $\Xi$  is fixed, after a simple manipulation with Lagrange multipliers, the minimum value of the function  $\text{vec}(\widehat{M} - \mathcal{M})'(\widehat{F}'\widehat{F} \otimes \widehat{F}'\widehat{F})\text{vec}(\widehat{M} - \mathcal{M})$  under the linear constraints (A.12) on  $\mathcal{M}$ , can be expressed as

$$\mathcal{F} = ((I_n \otimes \Xi)\text{vec}(\widehat{M}))'((I_n \otimes \Xi)(\widehat{F}'\widehat{F} \otimes \widehat{F}'\widehat{F})^{-1}(I_n \otimes \Xi)')^{-1}((I_n \otimes \Xi)\text{vec}(\widehat{M})).$$

By using the condition (A.13), we can simplify  $\mathcal{F}$  as  $\mathcal{F} = \text{vec}(\Xi\widehat{M})'((\widehat{F}'\widehat{F}) \otimes I_{n-r})\text{vec}(\Xi\widehat{M})$ . By using the formula  $\text{tr}\{ABCD\} = (\text{vec}(D))'(C' \otimes A)\text{vec}(B)$  (see Theorem 3 on p. 31 in Magnus and Neudecker (1999)), we can further rewrite  $\mathcal{F}$  as  $\mathcal{F} = \text{tr}\{\Xi\widehat{M}\widehat{F}'\widehat{F}\widehat{M}\Xi'\}$ . The MINCHI2 statistic can then be obtained by minimizing  $N\mathcal{F}$  under the constraint (A.13) on  $\Xi$ , that is,

$$\widehat{\xi}_{mcs}(r) = N \min_{\Xi(\widehat{F}'\widehat{F})^{-1}\Xi' = I_{n-r}} \text{tr}\{\Xi\widehat{M}\widehat{F}'\widehat{F}\widehat{M}\Xi'\} = N \min_{X'X = I_{n-r}} \text{tr}\{X'\widehat{F}\widehat{M}\widehat{F}'\widehat{M}X\},$$

where, in the last step, we made the change of variables  $X' = \Xi\widehat{F}^{-1}$ . Finally, by using the formula

$$\min_{X'X = I_k} \text{tr}\{X'AX\} = \sum_{i=1}^k \lambda_i,$$

where  $\lambda_1 \leq \dots \leq \lambda_n$  are the eigenvalues of an  $n \times n$  matrix  $A$  (see Theorem 13 on p. 211 in Magnus and Neudecker (1999)), we conclude that

$$\widehat{\xi}_{mcs}(r) = N \sum_{k=1}^{n-r} \widehat{\gamma}_k^2,$$

where  $0 \leq \widehat{\gamma}_1^2 \leq \dots \leq \widehat{\gamma}_n^2$  are the eigenvalues of the matrix  $\widehat{F}\widehat{M}\widehat{F}'\widehat{F}\widehat{M}\widehat{F}'$ . One can see that  $\widehat{\gamma}_k^2$ ,  $k = 1, \dots, n$ , are also the eigenvalues of the matrix  $\widehat{M}\widehat{F}'\widehat{F}\widehat{M}\widehat{F}'\widehat{F}$ , which yields the result.  $\square$

PROOF OF THEOREM 5.2: Suppose first that  $H_0$  holds. Restriction  $\text{rk}\{\mathcal{M}\} \leq r$  can be expressed as

$$\mathcal{M} = (\mathcal{M}_1 \ \mathcal{M}_1\Xi_1), \quad (\text{A.14})$$

where  $\mathcal{M}_1$  and  $\Xi_1$  are  $n \times r$  and  $r \times (n-r)$  matrices, respectively. Since the matrix  $\mathcal{M}$  is symmetric, there are  $(n-r)r + r(r+1)/2 =: s$  free parameters  $\mu$  for  $\mathcal{M}$  in (A.14) (these are the free parameters of  $\mathcal{M}_1$  since  $\Xi_1 = \Xi_1(\mathcal{M}_1)$  in the case of symmetric matrices) and hence

$$\widehat{\xi}_{mcs}(r) = N \min_{\mu} \text{vech}(\widehat{M} - \mathcal{M}(\mu))'\widehat{C}^{-1}\text{vech}(\widehat{M} - \mathcal{M}(\mu)), \quad (\text{A.15})$$

where  $\mathcal{M}(\mu) = (\mathcal{M}_1(\mu) \ \mathcal{M}_1(\mu)\Xi_1(\mu))$ . Let now  $B(\mu)$  be an  $n(n+1)/2 \times s$  matrix defined by  $B(\mu) = \partial \text{vec}(\mathcal{M}(\mu))/\partial \mu$ . Since  $\text{rk}\{M\} = r$ , we have  $M = \mathcal{M}(\mu_0)$  for some  $\mu_0$  and, moreover, the corresponding submatrix  $\mathcal{M}_1$  has full column rank. We obtain that the matrix  $B(\mu_0)$  is of full column rank, that is,  $\text{rk}\{B(\mu_0)\} = s$ .

Let  $\hat{\mu}$  be  $\mu$  minimizing the expression on the right-hand side of (A.15), that is,

$$\hat{\xi}_{mcs}(r) = N \text{vech}(\widehat{M} - \mathcal{M}(\hat{\mu}))' \widehat{C}^{-1} \text{vech}(\widehat{M} - \mathcal{M}(\hat{\mu})). \quad (\text{A.16})$$

We have  $\hat{\mu} \rightarrow_p \mu_0$ . Observe now that, by using the Taylor expansion and  $\hat{\mu} \rightarrow_p \mu_0$ , we have

$$\text{vech}(\widehat{M} - \mathcal{M}(\hat{\mu})) = \text{vech}(\widehat{M} - \mathcal{M}(\mu_0)) - B(\mu_0)(\hat{\mu} - \mu_0) + o_p(1). \quad (\text{A.17})$$

The first order conditions for minimizing (A.15), together with (A.17), imply that

$$0 = B(\hat{\mu})' \widehat{C}^{-1} \text{vech}(\widehat{M} - \mathcal{M}(\hat{\mu})) = B(\mu_0)' \widehat{C}^{-1} \text{vech}(\widehat{M} - \mathcal{M}(\mu_0)) - B(\mu_0)' \widehat{C}^{-1} B(\mu_0)(\hat{\mu} - \mu_0) + o_p(1)$$

and hence that

$$\hat{\mu} - \mu_0 = (B(\mu_0)' \widehat{C}^{-1} B(\mu_0))^{-1} B(\mu_0)' \widehat{C}^{-1} \text{vech}(\widehat{M} - \mathcal{M}(\mu_0)) + o_p(1). \quad (\text{A.18})$$

By substituting (A.18) into (A.17) and then (A.17) into (A.16), we get that

$$\begin{aligned} \hat{\xi}_{mcs}(r) &= N \text{vech}(\widehat{M} - \mathcal{M}(\mu_0))' (\widehat{C}^{-1/2})' \left( I_{n(n+1)/2} - \widehat{C}^{-1/2} B(\mu_0) (B(\mu_0)' \widehat{C}^{-1} B(\mu_0))^{-1} \right. \\ &\quad \left. \cdot B(\mu_0)' \widehat{C}^{-1/2} \right) \widehat{C}^{-1/2} \text{vech}(\widehat{M} - \mathcal{M}(\mu_0)) + o_p(1) \\ &= (\sqrt{N} C^{-1/2} \text{vech}(\widehat{M} - M))' (I_{n(n+1)/2} - A_0 (A_0' A_0)^{-1} A_0') (\sqrt{N} C^{-1/2} \text{vech}(\widehat{M} - M)) + o_p(1), \end{aligned}$$

where  $A_0 = C^{-1/2} B(\mu_0)$ . By (2.10) and since the matrix  $B(\mu_0)$  or the matrix  $A_0$  has full column rank, we obtain that

$$\hat{\xi}_{mcs}(r) \xrightarrow{d} \chi^2(n(n+1)/2 - \text{rk}\{B(\mu_0)\}) = \chi^2((n-r)(n-r+1)/2).$$

The proof under  $H_1$  follows since

$$\hat{\xi}_{mcs}(r) N^{-1} \xrightarrow{p} \min_{\text{rk}\{\mathcal{M}\} \leq r, \mathcal{M} = \mathcal{M}'} \text{vech}(M - \mathcal{M})' C^{-1} \text{vech}(M - \mathcal{M}) > 0$$

for  $r < \text{rk}(M)$  and  $N \rightarrow \infty$ .  $\square$

**PROOF OF THEOREM 6.1:** We only consider the more difficult case of  $H_0 : \text{rk}\{M\} = r$ . Suppose first that the permutations  $R_k$  in (6.4) are identity and that they are defined uniquely (that is, there are no ties). Then, for large  $N$ ,  $\widehat{R}_k = I_n$ ,  $\widehat{k}(r) = k(r)$  and  $\widehat{l}(r) = l(r) = r$ . As in Cragg and Donald (1996), p. 1308, we have

$$\widehat{m}_{22}(\widehat{k}(r)) = \widehat{M}_{22} - \widehat{M}_{21} \widehat{M}_{11}^{-1} \widehat{M}_{12} = \Phi(\widehat{M} - M) \Phi' +_p(N^{-1}), \quad (\text{A.19})$$

where  $\Phi = (-M_{21} M_{11}^{-1} \ I_{n-r})$  and  $M = (M_{11} M_{12}; M_{21} M_{22})$  is a partition with  $r \times r$  submatrix  $M_{11}$ . Relation (A.19) implies that

$$\text{vech}(\widehat{m}_{22}(\widehat{k}(r))) = \Pi_s \text{vech}(\widehat{M} - M) + O_p(N^{-1}), \quad (\text{A.20})$$

where  $\Pi_s = D_{n-r}^+(\Phi \otimes \Phi)D_n$ . By using Assumptions  $(A^*)$ , we obtain that

$$\sqrt{N}\text{vech}\left(\widehat{m}_{22}(\widehat{k}(r))\right) \xrightarrow{d} \mathcal{N}(0, \Pi_s C \Pi_s')$$

and hence  $\widehat{\xi}_{sldu}(r) \rightarrow_d \chi^2((n-r)(n-r+1)/2)$ . When the permutations  $R_k$  in (6.4) are not identity but still defined uniquely, we have  $\widehat{R}_k = R_k$  for large  $N$  and the matrices  $\widehat{M}$  and  $M$  are permuted in advance so that permutations become no longer necessary.

The case of ties can be dealt with as in Cragg and Donald (1996). Let  $\widehat{\xi}_{sldu}^a(r)$  be defined by using one set of possible permutations  $\widehat{R}_k^a$  indexed by  $a$  and satisfying  $\widehat{R}_k^a = R_k^a$  for large  $N$ . Taking into account the permutations  $\widehat{R}_k^a$  beforehand,  $\widehat{\xi}_{sldu}^a(r)$  is thus defined in terms of the matrix  $\widehat{M}_a = \widehat{P}_a \widehat{M} \widehat{P}_a'$  where  $\widehat{P}_a$  is a permutation matrix, and the matrix  $\widehat{C}_a$  which is a consistent estimator for the limit covariance matrix  $C_a$  appearing in  $\sqrt{N}\text{vech}(\widehat{M}_a - M_a) \rightarrow_d \mathcal{N}(0, C_a)$ . Let also

$$\widehat{\xi}_{mcs}(r) = N \min_{\text{rk}\{\mathcal{M}\}=r} \text{vec}(\widehat{M} - \mathcal{M})' D_n^{+'} \widehat{C}^{-1} D_n^+ \text{vec}(\widehat{M} - \mathcal{M}). \quad (\text{A.21})$$

To show that ties make no difference asymptotically to  $\widehat{\xi}_{sldu}$ , it is enough to prove that  $\widehat{\xi}_{sldu}^a - \widehat{\xi}_{mcs} \rightarrow_p 0$ .

One can verify that, for any  $a$ ,

$$\widehat{\xi}_{mcs}(r) = N \min_{\text{rk}\{\mathcal{M}\}=r} \text{vec}(\widehat{M}_a - \mathcal{M})' D_n^{+'} \widehat{C}_a^{-1} D_n^+ \text{vec}(\widehat{M}_a - \mathcal{M}). \quad (\text{A.22})$$

Since  $\widehat{M}_a$  is defined by permuting the matrix  $\widehat{M}$  for the Gaussian elimination, the restriction in the minimum of (A.22) can be replaced by  $\mathcal{M} = (\mathcal{M}_{11} \mathcal{M}_{12}; \mathcal{M}_{21} \mathcal{M}_{22})$  such that  $\mathcal{M}_{22} = \mathcal{M}_{21} \mathcal{M}_{11}^{-1} \mathcal{M}_{12}$  and  $\mathcal{M}_{11}$  is  $r \times r$ . As in Cragg and Donald (1996), p. 1309, we have

$$\begin{aligned} \widehat{\xi}_{mcs}(r) &= N \text{vec}(\widehat{M}_a - M_a)' \left( D_n^{+'} \widehat{C}_a^{-1} D_n^+ - D_n^{+'} C_a^{-1} D_n^+ B (B' D_n^{+'} C_a^{-1} D_n^+ B)^{-1} B' D_n^{+'} C_a^{-1} D_n^+ \right) \\ &\quad \cdot \text{vec}(\widehat{M}_a - M_a) + o_p(1) = N (C_a^{-1/2} \text{vech}(\widehat{M}_a - M_a))' \\ &\quad \cdot \left( I_{n(n+1)/2} - C_a^{-1/2} D_n^+ B (B' D_n^{+'} \widehat{C}_a^{-1} D_n^+ B)^{-1} B' D_n^{+'} \widehat{C}_a^{-1/2} \right) (C_a^{-1/2} \text{vech}(\widehat{M}_a - M_a)) + o_p(1), \end{aligned}$$

where  $B = \partial \text{vec}(\mathcal{M}) / \partial \text{vec}(\phi)'$  and  $\phi' = (\text{vec}(\mathcal{M}_{11})', \text{vec}(\mathcal{M}_{21})', \text{vec}(\mathcal{M}_{12})')$  are the free parameters. Similarly,

$$\widehat{\xi}_{sldu}^a(r) = N (C_a^{-1/2} \text{vech}(\widehat{M}_a - M_a))' \left( C_a^{1/2} \Pi_s^{a'} (\Pi_s^a C_a \Pi_s^{a'})^{-1} \Pi_s^a C_a^{1/2} \right) (C_a^{-1/2} \text{vec}(\widehat{M}_a - M_a)) + o_p(1),$$

where  $\Pi_s^a = D_{n-r}^+(\Phi_a \otimes \Phi_a)D_n$  with  $\Phi_a = (-M_{a,21} M_{a,11}^{-1} \ I_{n-r})$ . It is now enough to show that

$$I_{n(n+1)/2} - C_a^{-1/2} D_n^+ B (B' D_n^{+'} C_a^{-1} D_n^+ B)^{-1} B' D_n^{+'} C_a^{-1/2} = C_a^{1/2} \Pi_s^{a'} (\Pi_s^a C_a \Pi_s^{a'})^{-1} \Pi_s^a C_a^{1/2}.$$

This can be proved as in Cragg and Donald (1996), p. 1309, as long as  $\Pi_s^a D_n^+ B = 0$ . The latter follows from the identity

$$\Pi_s^a D_n^+ B = D_{n-r}^+(\Phi_a \otimes \Phi_a)D_n D_n^+ B = D_{n-r}^+(\Phi_a \otimes \Phi_a)B = 0,$$

where we used  $(\Phi_a \otimes \Phi_a)D_n D_n^+ = D_{n-r}^+(\Phi_a \otimes \Phi_a)$  (see Theorem 12,(b), p. 49, and Theorem 9,(a), p. 47, in Magnus and Neudecker (1999)), and the fact that  $(\Phi_a \otimes \Phi_a)B = 0$  as proved in Cragg and Donald (1996), p. 1309.  $\square$

PROOF OF PROPOSITION 7.1: We can rewrite (7.1) as

$$M = U\Upsilon U'$$

with a diagonal matrix  $\Upsilon$  such that  $\text{abs}(\Upsilon) = S$ . Then,  $V = UQ$  with  $Q = \text{sign}(\Upsilon)$ . Partition  $Q$  as in (7.2) into  $Q = (Q_1 \ 0; 0 \ Q_2)$  so that  $V_{12} = U_{12}Q_2$ ,  $V_{22} = U_{22}Q_2$ . Partition also  $\Upsilon$  into  $\Upsilon = (\Upsilon_1 \ 0; 0 \ \Upsilon_2)$  so that  $\Upsilon_2 = S_2Q = S_2Q'_2$ . Then,

$$B'_{r,\perp} = \begin{pmatrix} V_{12} \\ V_{22} \end{pmatrix} V_{22}^{-1} (V_{22}V'_{22})^{1/2} = \begin{pmatrix} U_{12} \\ U_{22} \end{pmatrix} Q_2 Q_2^{-1} U_{22}^{-1} (U_{22}Q_2Q'_2U'_{22})^{1/2} = A_{r,\perp},$$

since  $Q_2^{-1} = Q'_2$ . Similarly,

$$\Lambda_r = (U_{22}U'_{22})^{-1/2} U_{22} S_2 V_{22} (V_{22}V'_{22})^{-1/2} = U_{22}U'_{22})^{-1/2} U_{22} \Upsilon'_2 V'_{22} (V_{22}V'_{22})^{-1/2}$$

showing that  $\Lambda_r$  is symmetric. Analogous results hold for  $\hat{\Lambda}_r$ ,  $\hat{A}_{r,\perp}$  and  $\hat{B}_{r,\perp}$ . Asymptotic normality of  $\hat{\Lambda}_r$  has been proved in Theorem 1 of Kleibergen and Paap (2003).  $\square$

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Homoscedasticity assumptions						
Statistic	$h$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$
EIG	0.1	0.0000	0.0000	0.5521	0.7964	0.8226
	0.3	0.0000	0.0000	0.9183	0.9796	0.9571
	0.45	0.0000	0.0000	0.9621	0.9859	0.9504
	0.55	0.0000	0.0000	0.9666	0.9831	0.9515
MINCHI2	0.1	0.0000	0.0000	0.9363	0.8213	0.8052
	0.3	0.0000	0.0000	0.6014	0.6571	0.6323
	0.45	0.0000	0.0000	0.5647	0.5475	0.9767
	0.55	0.0000	0.0000	0.5976	0.6079	0.8639
LDU	0.1	0.0000	0.0000	0.9260	0.9963	0.8091
	0.3	0.0000	0.0000	0.6196	0.3916	0.7088
	0.45	0.0000	0.0000	0.6002	0.5589	0.9771
	0.55	0.0000	0.0000	0.6383	0.4892	0.8471
SVD	0.1	0.0000	0.0000	0.9363	0.8213	0.8052
	0.3	0.0000	0.0000	0.6014	0.6571	0.6323
	0.45	0.0000	0.0000	0.5647	0.5475	0.9767
	0.55	0.0000	0.0000	0.5976	0.6079	0.8639

Table 1:  $p$ -values in rank estimation

Heteroskedasticity assumptions						
Statistic	$h$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$
EIG	0.1	0.0000	0.0003	0.4528	0.8336	0.8322
	0.3	0.0000	0.0000	0.7443	0.8837	0.8744
	0.45	0.0000	0.0000	0.9355	0.9374	0.9217
	0.55	0.0000	0.0000	0.9270	0.9279	0.9262
MINCHI2	0.1	—	—	—	—	—
	0.3	—	—	—	—	—
	0.45	—	—	—	—	—
	0.55	—	—	—	—	—
LDU	0.1	0.0000	0.0000	0.9084	0.9961	0.8059
	0.3	0.0000	0.0000	0.5983	0.3914	0.7099
	0.45	0.0000	0.0000	0.5606	0.5420	0.9773
	0.55	0.0000	0.0000	0.6151	0.4809	0.8485
SVD	0.1	0.0000	0.0000	0.8867	0.7963	0.8054
	0.3	0.0000	0.0000	0.6034	0.4113	0.6162
	0.45	0.0000	0.0000	0.5652	0.5835	0.9772
	0.55	0.0000	0.0000	0.6091	0.6056	0.8502

Table 2:  $p$ -values in rank estimation

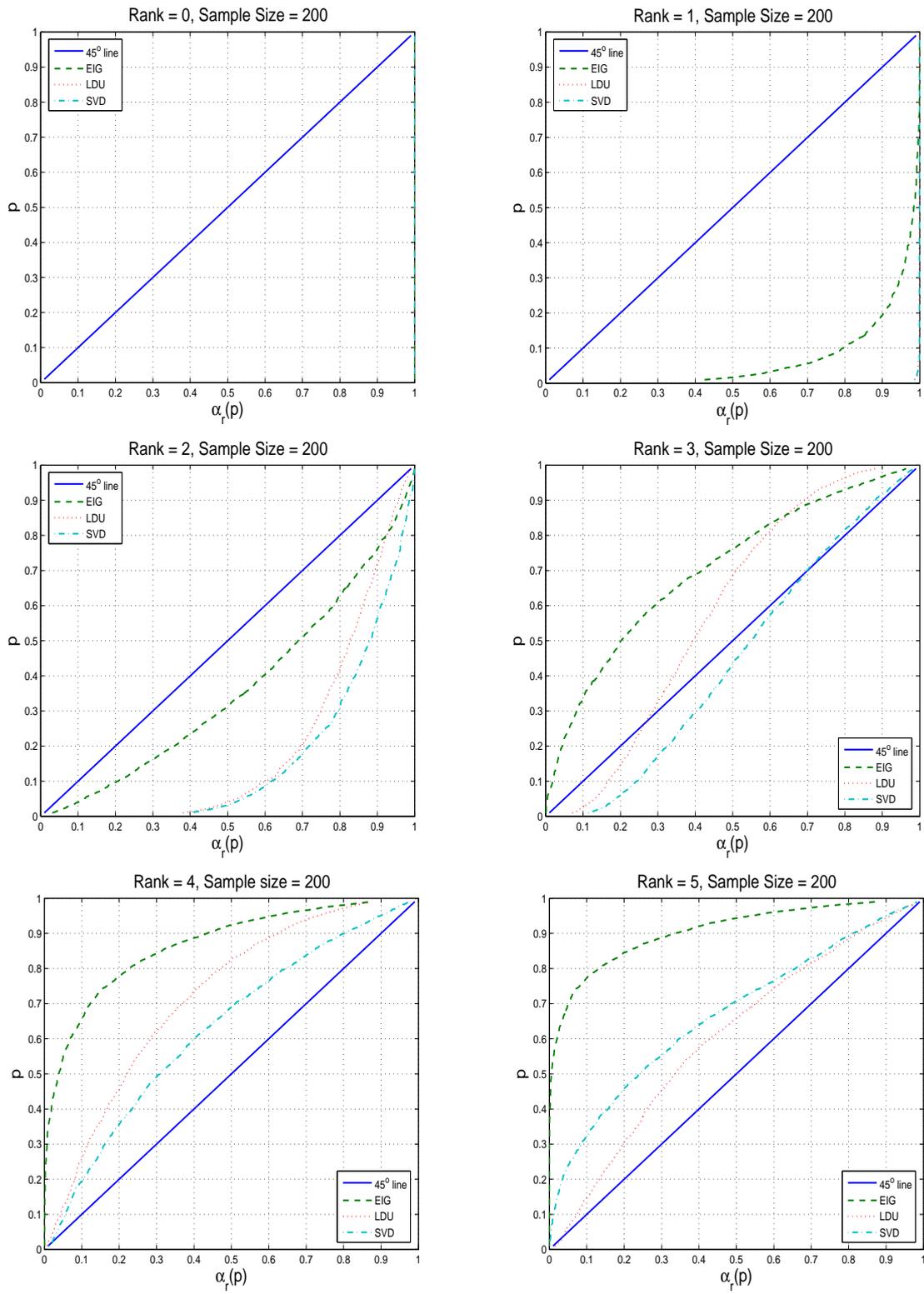


Figure 1: PP-plots in a simulation study.

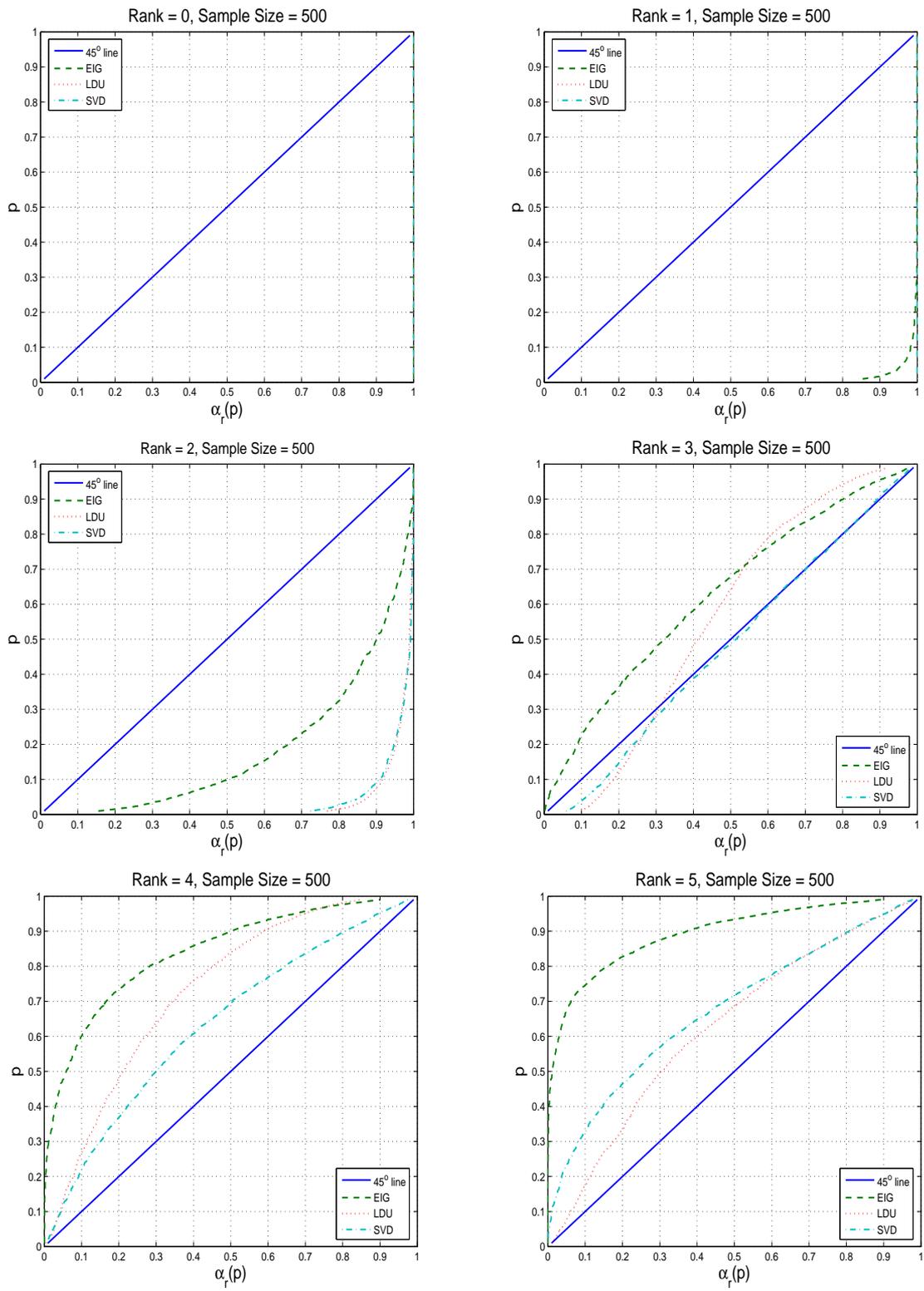


Figure 2: PP-plots in a simulation study.

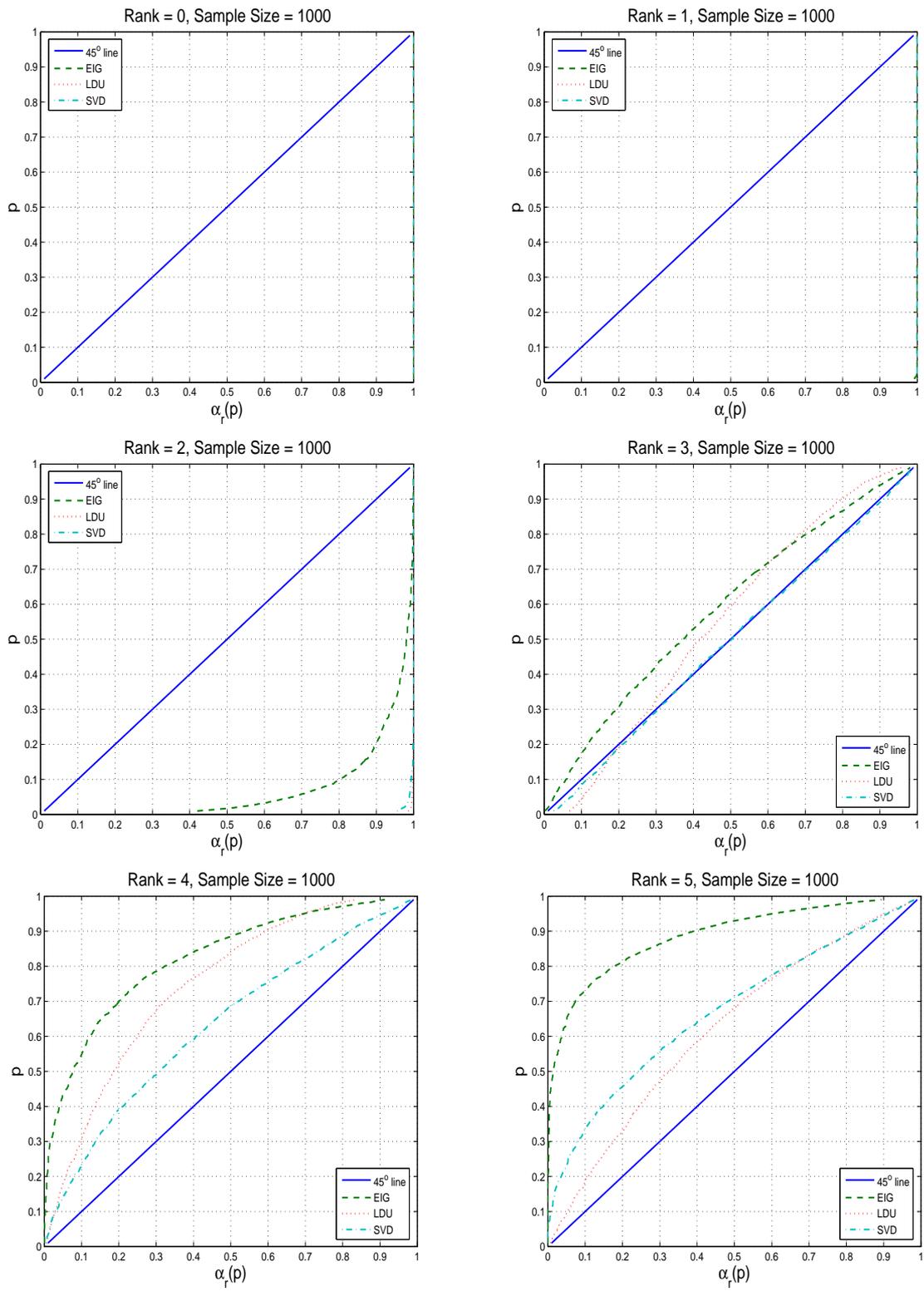


Figure 3: PP-plots in a simulation study.

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